DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

May 5, 2000

Questions have equal value, but within a given question not all parts have equal value. Proof and justifications are expected unless the question specifies otherwise.
You may take the fundamental group and homology groups and cohomology ring of spheres as "known" and quote them without proof wherever needed.

1. The *finite* topology on a set X is the topology whose closed sets are X and its finite subsets.

(a) Define the terms "separable", "Hausdorff", and "second countable".

(b) Show that the finite topology on $\mathbb{R}$ is separable but neither Hausdorff nor second countable.

(c) Consider the identity map between $\mathbb{R}$ with the standard topology and $\mathbb{R}$ with the finite topology. Is it continuous? Is it continuous in the reverse direction?

2. (a) Define "CW-complex".

(b) How does the definition simplify in the case of a CW-complex with only finitely many cells?

(c) Why must a CW-complex with only finitely many cells be compact?

(d) State whether or not the converse is true. That is, must a compact CW-complex have only finitely many cells? (Justification for your answer is not required).

3. (a) Let x and y be points in a path connected space X. Describe the relationship between $\pi_1(X, x)$ and $\pi_1(X, y)$.

(b) Let $X = S^1 \vee S^1$. Show by example that it is possible for homotopic maps (which preserve the basepoint) f and g from S^1 to X to represent different elements of $\pi_1(X)$ if the homotopy is not required to be basepoint preserving. What property of $\pi_1(X)$ is it that makes such an example possible?

4. Let X be the space formed by taking two Möbius bands and "gluing them together" by identifying their boundaries.

(a) The description of X appearing above is rather informal. Give a mathematical definition of what "gluing them together" means.

(b) Calculate the homology groups of X.

(c) Explain briefly why X is a manifold and identify it. (Two or three explanatory sentences is all that is expected; a detailed justification is not required.)

(d) Is X orientable?

5. Let T denote the (standard) 2-dimensional torus.

(a) State the homology and cohomology of T including the ring structure. (Just state the results; no justification is required.)

(b) State the fundamental group of T with a brief explanation of how you arrived at this answer. (Detailed proof is not required.)

(c) Show that every map from the sphere S^2 to T induces the zero map on cohomology.

6. (a) Let $p : E \to B$ be a covering projection between path connected spaces. Show that p induces an injection on π_1.

(b) Deduce that the only connected covering space of a simply connected space is itself. (That is, if B is simply connected $p : E \to B$ is a homeomorphism).

(c) Show that a simply connected manifold must be orientable.