DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Friday, May 10, 2002, 1–4 p.m.

Questions have equal value, but different parts of a question may have different weights.

1. a) Give the definitions for *connected space* and for *path-connected space.*

b) Give an example of a topological space X that is connected but not path connected.

c) Give the definitions for *Hausdorff space* and for *normal space.*

d) Prove that every compact Hausdorff space is normal.

2. a) State the Mayer-Vietoris theorem for the homology of a topological space X.

b) Use the theorem to compute the homology groups of S^n for all n.

c) Use the theorem to compute the homology groups of the real projective plane $\mathbb{R}P(2)$.

3. a) Let $f : S^n \to S^n$ be a continuous map, $n \geq 1$. State the definition of the degree of this map.

b) What is the degree of the antipodal map

$$a : S^n \to S^n, \quad a(x) = -x,$$

and what is the degree of the identity map? (Give a brief justification of your answer.)

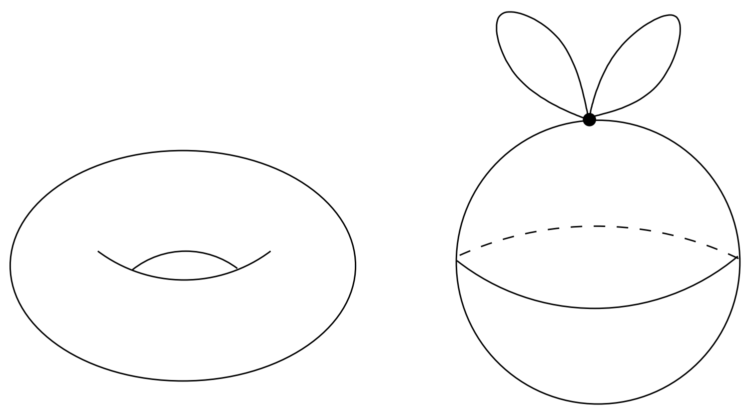
c) Show that if $h : S^n \to S^n$ has degree different from that of the antipodal map, then h has a fixed point.

4. Let X be a topological space, and R a commutative ring.

a) State the formula for the coboundary $\delta(c_1 \cup c_2)$ of the cup product of two singular cochains c_1, c_2 in X, with coefficients in R.

b) Using a), show that the cup product on cochains induces a ring structure on the cohomology with coefficients in R.

c) State the cohomology groups with coefficients in $\mathbb{Z}$ of the 2-torus $X_1 = T^2$ and of the space $X_2 = S^1 \vee S^1 \vee S^2$.



X_1 X_2

Compare the ring structure on the cohomology of X_1, X_2. (You are not asked to justify your answer.)

5. a) State van Kampen's theorem.

b) Show that if X is a topological n-manifold with $n \geq 3$, the fundamental group $\pi_1(X)$ is unchanged by removal of an n-dimensional disk from X.

c) Let X_1, X_2 be two connected, oriented topological n-manifolds, with $n \geq 3$. The connected sum $X_1 \# X_2$ is defined by deleting two disks $D_j \subset X_j$ and identifying the boundaries $\partial(X_j \backslash D_j) \cong S^{n-1}$ by an orientation-reversing homeomorphism. Express the fundamental group of $X_1 \# X_2$ in terms of $\pi_1(X_1), \pi_1(X_2)$.

6. a) Give the definition of a strong deformation retraction of a topological space X.

b) Find the first fundamental group of the following spaces. (Give a brief justification of your answer.) Which of these groups are abelian?

(1) S^2 with two points removed,

(2) Klein bottle with a point removed,

(3) Torus with two points removed.