DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Monday, September 9, 2002, 1–4 p.m.

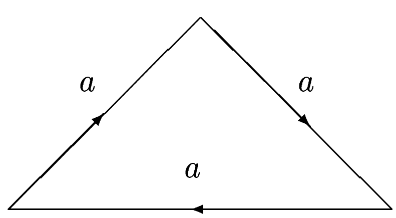
Questions have equal value, but different parts of a question may have different weights.

1. a) Define "topological group".

b) Prove that if $f : X \to Y$ is continuous and K is a compact subset of X, then $f(K)$ is a compact subset of Y.

c) Let A and B be compact subsets of a topological group G. Show that AB is compact (where AB denotes the subset $\{x \in G \mid x = ab \text{ for some } a \in A, b \in B\}$).

2. Let Y denote the space formed by taking a triangle (including interior) and identifying boundary sides as shown in the following picture.



a) State Van Kampen's theorem in full, and use it to show that $\pi_1(Y) = \mathbb{Z}/3\mathbb{Z}$.

b) Use the standard covering projection $p : S^2 \to \mathbb{R}P^2$ (where $\mathbb{R}P^2$ denotes the real projective plane) to show that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2\mathbb{Z}$.

3. a) Let $p : X \to B$ be covering projection and let $f : Y \to B$ be a continuous map. State in full the lifting theorem, which gives necessary and sufficient conditions for the existence and uniqueness of a lift of f to a map $\tilde{f} : Y \to X$ such that $f = p \circ \tilde{f}$.

b) Let $p : S^2 \to \mathbb{R}P^2$ denote the covering projection and let Y be as in question 2. Is it true that every map from Y to $\mathbb{R}P^2$ can be lifted to S^2?

4. a) Show that $\mathbb{C}P^n$ (where $\mathbb{C}P^n$ denotes complex projective n-space) has a CW structure (i.e. cell decomposition) with precisely one cell in every even dimension and no odd dimensional cells.

b) How does such a cell decomposition tell you what the homology groups of $\mathbb{C}P^n$ are?

c) Prove that there cannot be a cell decomposition of $\mathbb{C}P^n$ with fewer cells than the cell decomposition in the one from part (a).

5. Let T denote the (standard) 2-dimensional torus. Calculate the cohomology of T including the ring (cup product) structure.

6. a) Define the "degree" of a continuous self-map of a sphere.

b) What is the degree of the identity map? What is the degree of a reflection $r(x_0, x_1, \ldots, x_k) = (-x_0, x_1, \ldots, x_k)$? What is the degree of the antipodal map $a(x) = -x$?

c) Given a nowhere zero vector field, use it to describe a homotopy from the identity map to the antipodal map.

d) Prove that if n is even then S^n does not have a nowhere zero vector field. (That is, show that there does not exist a continuous function which associates to each $x \in S^n$ a nonzero tangent vector to x.)