DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Friday, May 9, 2003, 1–4 p.m.

1. A sequence $(x_i)_{i=1}^{\infty}$ of points in a topological space X is said to converge to $x \in X$ if for every neighbourhood U of x there exists N such that $x_i \in U$ for all $i \geq N$.

(a) Show by example that it is possible for a sequence to converge to more than one limit.

(b) Show that if X is Hausdorff then any sequence in X has at most one limit.

(c) Prove or disprove: Any sequence $(x_i)_{i=1}^{\infty}$ of points in a compact Hausdorff space X has a convergent subsequence.

(d) Prove or disprove: Suppose X, Y are Hausdorff spaces, and $f : X \to Y$ is a continuous map. Suppose the sequence $(x_i)_{i=1}^{\infty}$ of points in X converges. Then the sequence $(f(x_i))_{i=1}^{\infty}$ of image points converges.

2. Prove: A topological space X is Hausdorff if and only if the diagonal $\Delta_X \subset X \times X$ is a closed subset of X.

3. Let $D = \{z \in \mathbb{C} | \, |z| \leq 1\}$ and $S^1 = \{z \in \mathbb{C} | \, |z| = 1\}$. Consider the map

$$f : S^1 \to S^1 \times S^1, \; z \mapsto (z^2, z^3),$$

and let $X = ((S^1 \times S^1) \cup D)/\sim$ be the space obtained by identifying $z \in S^1 = \partial D$ with $f(z) \in S^1 \times S^1$. Find the fundamental group $\pi_1(X)$.

4. Suppose X is a connected, locally path connected space with finite fundamental group. Show that any continuous map $f : X \to S^1$ is homotopic to the constant map.

5. (a) State the Mayer-Vietoris theorem.

(b) An open cover $\mathcal{U} = \{U_a\}$ of a topological space X is called *good* if each finite non-empty intersection $U_{a_1} \cap \cdots \cap U_{a_k}$ of elements of this cover is contractible. Use induction on n to show that if X admits a good open cover by n open sets, then $H_i(X) = 0$ for $i > n - 1$.

6. Let M be a compact oriented n-manifold with $n > 0$. Show that M cannot be contractible. Does this generalize to the non-orientable case?