DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Monday, September 8, 2003, 1–4 p.m.

Questions have equal value, but different parts of a question may have different weights.

In any question, you may use without proof your knowledge of the homology groups $H_\bullet(S^k;\Gamma)$ for any integer k and any coefficient group Γ.

1. The *finite topology* on a set X is the topology for which the non-empty open sets are sets of the form $U = X\backslash F$ with F a finite subset.

(a) Define the terms "Hausdorff" and "separable".

(b) Prove or disprove:

(i) The finite topology on $X = \mathbb{R}$ is Hausdorff.
(ii) The finite topology on $X = \mathbb{R}$ is separable.
(iii) The identity map from $\mathbb{R}$, with the standard topology, to $\mathbb{R}$, with the finite topology, is continuous.

2. Let X be the space formed by the edges of the cube $[0,1]^3 \subset \mathbb{R}^3$. Calculate the fundamental group $\pi_1(X)$.

3. Let X be the wedge product of the projective plane with the 2-sphere, $X = S^2 \vee \mathbb{R}P^2$.

(a) Describe the universal covering space $\tilde{X}$ of X.

(b) What is the definition of "deck" (or "covering") transformation?

(c) For the above space X, describe the group of deck transformations, and explain how it acts on $\tilde{X}$.

4. Let Γ be an Abelian group, and X a topological space.

(a) State the Mayer-Vietoris theorem for $H_\bullet(X;\Gamma)$, corresponding to a covering of X by two open subsets $U, V \subset X$.

(b) Calculate the homology groups $H_\bullet(\mathbb{R}P^2; \Gamma)$ for the following two cases:

(i) $\Gamma = \mathbb{Z}_3$
(ii) $\Gamma = \mathbb{Z}_4$

Hint: Identify $\mathbb{R}P^2 = D^2/\sim$ (antipodal identification on the boundary ∂D^2), and take $U,\ V$ to be the images of $D^2 \backslash \partial D^2,\ D^2 \backslash \{0\}$ under the quotient map.

5. Calculate the homology groups of the space $X = S^2 \times S^1$ with coefficients in $\mathbb{Z}$.

6. (a) State the Poincaré duality theorem for the cohomology groups of compact oriented manifolds.

(b) Let $f : M \to N$ be a continuous map between compact oriented manifolds of dimension n, and assume that the induced map $f^* : H^n(N) \to H^n(M)$ is an ismorphism. Show that $f^* : H^p(N) \to H^p(M)$ is injective for all p.