DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Monday, May 3, 2004, 1-4 PM

The following 5 questions have equal value, but different parts of a question may have different weights.

1. Prove: a continuous real valued function on a compact topological space is bounded and attains its maximal value.

2. Describe the fundamental groups and the universal covers of the following spaces:
   (a) The torus $T^2 = S^1 \times S^1$.
   (b) The punctured torus $X = T^2 - \{p\}$ where $p \in T^2$ is a single point.
   (c) The real projective plane $\mathbb{RP}^2$.
   (d) The punctured projective plane $\mathbb{RP}^2 - \{p\}$ where $p \in \mathbb{RP}^2$ is a single point.

3. (a) Let X be the union of two open subsets U and V. State the Mayer-Vietoris theorem relating the homology groups of X, U, V and $U \cap V$.
   (b) Two solid tori $D^2 \times S^1$ are glued along their boundaries using the gluing map $f : T^2 \to T^2$ corresponding to the integer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. (Such a matrix defines a map $\mathbb{R}^2 \to \mathbb{R}^2$ which descends to the quotient $T^2 = \mathbb{R}^2/\mathbb{Z}^2$). Compute the homology of the resulting space L.

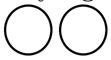
4. (a) Define "the Euler characteristic $\chi(X)$ of a topological space X" (did you need to make any assumptions so that the definition would make sense?).
   (b) Prove that if X_1 and X_2 are homotopy equivalent then $\chi(X_1) = \chi(X_2)$.
   (c) Prove that the Euler characteristic of an odd-dimensional compact oriented manifold is always 0.

5. (a) Define the *degree* $\deg \Phi$ of a continuous map $\Phi : T^2 \to S^2$.

(b) Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ be two continuous maps such that $\gamma_1(S^1) \cap \gamma_2(S^1) = \emptyset$. Define $\Phi_{(\gamma_1,\gamma_2)} : T^2 = S^1 \times S^1 \to S^2$ by

$$\Phi_{(\gamma_1,\gamma_2)}(z_1, z_2) := \frac{\gamma_2(z_2) - \gamma_1(z_1)}{|\gamma_2(z_2) - \gamma_1(z_1)|},$$

for $z_1, z_2 \in S^1$. Prove that the degree, $l(\gamma_1, \gamma_2) := \deg \Phi_{(\gamma_1,\gamma_2)}$, is invariant under homotopies $\gamma_{1,t}, \gamma_{2,t}$ of γ_1, γ_2 for which $\gamma_{1,t}(S^1) \cap \gamma_{2,t}(S^1) = \emptyset$ for all t.

(c) Compute (without worrying about signs) $l(\gamma_1, \gamma_2)$ for the embedding of two circles given by the picture .

(d) Compute (without worrying about signs) $l(\gamma_1, \gamma_2)$ for the embedding of two circles given by the picture .

Good Luck!