DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Monday, September 13, 2004, 1-4 PM

The following 6 questions have equal value, but different parts of a question may have different weights.

1. (a) Let X be a compact metric space and let $\{U_\alpha : \alpha \in A\}$ be an open cover of X. Show that there exists $\epsilon > 0$ such that for every $x \in X$ there exists $\alpha \in A$ such that the ϵ-ball centred at X is contained in U_α. (ϵ is called a *Lebesgue number* for the covering.)

   (b) Prove that any continuous function on a compact metric space is uniformly continuous.

2. A *topological group* is a group G which is also a topological space, so that the multiplication map $m : G \times G \to G$ and the inverse map $(x \mapsto x^{-1})$: $(G - \{e\}) \to (G - \{e\})$ are both continuous. Show that the fundamental group of a topological group G is Abelian (even if G is not!).

3. Let X be the *comb space* defined by

$$X = \left(\bigcup_{n=1}^{\infty} \left\{ \left(\frac{1}{n}, y \right) : 0 \le y \le 1 \right\} \right) \cup \{(0, y) : 0 \le y \le 1\} \cup \{(x, 0) : 0 \le x \le 1\} \subset \mathbb{R}^2.$$

   Let $I = \{(0, y) : 0 \le y \le 1\} \subset X$.

   (a) Sketch X.

   (b) Define *deformation retract* and *strong deformation retract.*

   (c) Show that I is a deformation retract of X.

   (d) Show that I is not a strong deformation retract of X.

4. Let $X = S^2 \cup C$ where $C = \{(0,0,z) : -1 \leq z \leq 1\}$ is the chord joining the south pole to the north pole.

   (a) Describe the universal covering of X (possibly using a picture to help with the description).

   (b) Compute $\pi_1(X)$.

5. Let S be the 2-dimensional skeleton of the four dimensional cube:

$$S = \big\{(x_i) \in [0,1]^4 \text{: for at least two } i\text{'s, } x_i = 0 \text{ or } x_i = 1\big\}$$

   (a) Compute the Euler characteristic of S.

   (b) Compute all homology groups of S.

6. (a) Prove: there is no continuous function $r : D^n \to S^{n-1}$ from the n-dimensional unit ball to its boundary, whose restriction to the boundary is the identity.

   (b) Prove: every continuous function $f : D^n \to D^n$ has a fixed point.

Good Luck!