DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Friday, May 6, 2005, 1-4 PM

1. Let $X_1 \subset X_2 \subset \cdots$ be a sequence of Hausdorff spaces where X_i is a closed subspace of X_{i+1} for each i. Let $X = \bigcup_{i=1}^{\infty} X_i$. Define the coherent topology on X by $U \subset X$ is open $\Leftrightarrow U \cap X_i$ is open in X_i $\forall i$.

a) Verify that this is a topology.

b) Show that X_i is a subspace of X in this topology.

c) Suppose that each X_i is normal; state Tietze's extension theorem and use it to show that X is normal.

2. a) Suppose $n \geq 2$. Does there exist a continuous map $f : S^n \to S^1$ which is not homotopic to a constant?

b) Suppose $n \geq 2$. Does there exist a continuous map $f : \mathbf{R}P^n \to S^1$ which is not homotopic to a constant?

c) Let $T = S^1 \times S^1$ be the torus. Does there exist a continuous map $f : T \to S^1$ which is not homotopic to a constant?

In each case, carefully justify your answer.

3. a) State $\pi_1(S^1)$. Say briefly what goes into showing this. (A proof is not required.)

b) Let sq: $S^1 \to S^1$ be the map $z \mapsto z^2$ where the circle is regarded as the unit ball of $\mathbf{C}$ and the multiplication is that from $\mathbf{C}$. Compute the group homomorphism $\mathrm{sq}_\sharp \colon \pi_1(S^1) \to \pi_1(S^1)$.

4. a) Give the definition of a strong deformation retraction of a topological space X.

b) Find the first fundamental group of the following spaces. (Give a brief justification of your answer.) Which of these groups are abelian?

(i) S^2 with two points removed,

(ii) Klein bottle with a point removed,

(iii) Torus with two points removed.

5. Calculate $H_*(\vee_k S^n, \mathbf{Z})$ for $k, n \geq 1$. (You may take $H_*(S^n)$ as known.)

6. a) Let $f : S^n \to S^n$ be a continuous map, $n \geq 1$. State the definition of the degree of this map.

b) What is the degree of the antipodal map

$$a : S^n \to S^n, \quad a(x) = -x,$$

and what is the degree of the identity map? (Give a brief justification of your answer.)

c) Show that if $h : S^n \to S^n$ has degree different from that of the antipodal map, then h has a fixed point.