DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Monday, September 12, 2005, 1-4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Good Luck!

1. The *finite* topology on a set X is the topology whose closed sets are the finite subsets of X. Verify that $\mathbb{C}$ with the finite topology is not Hausdorff, is not second countable, but is separable. Show that a polynomial with complex coefficients defines a continuous map from $\mathbb{C}$ to itself.

2. Let X be a compact metric space with metric d and let $f : X \to X$ be a contraction map; a map satifying $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$.

   (a) Prove that f is continuous or find a counterexample.

   (b) Prove that f has a fixed point; a point $x \in X$ for which $f(x) = x$.

3. Describe the fundamental groups and the universal covers of the following spaces:

   (a) The torus $T^2 = S^1 \times S^1$.

   (b) The punctured torus $X = T^2 - \{p\}$ where $p \in T^2$ is a single point.

   (c) The real projective plane $\mathbb{RP}^2$.

   (d) The punctured projective plane $\mathbb{RP}^2 - \{p\}$ where $p \in \mathbb{RP}^2$ is a single point.

4. Let $T^2 := S^1 \times S^1$ be the 2-torus. Let $X = S^1 \times \{1\} \cup \{1\} \times S^1 \subset T^2$. And let $i : X \to T^2$ denote the inclusion map.

   (a) Prove that $i_* : H_1(X) \to H_1(T^2)$ is an isomorphism.

   (b) Prove that $i_* : \pi_1(X) \to \pi_1(T^2)$ is not an isomorphism. What is the kernel of i_*?

5. (a) Let $f, g : S^n \to S^n$ be continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Show that $f \simeq a \circ g$, where a is the antipodal map.

   (b) Prove that any continuous map $f : S^{2n} \to S^{2n}$ either has a fixed point or there is a point x with $f(x) = -x$.

   (c) Prove that any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

6. Let $f : X \to Y$. Define the mapping cylinder M_f by $M_f := (X \times I) \cup Y/\sim$ where $(x, 1) \sim f(x)$ for all $x \in X$. Define the mapping cone C_f by $C_f := M_f/\sim$ where $(x, 0) \sim (x', 0)$ for all $x, x' \in X$.

   (a) Show that there is a long exact homology sequence

   $$\cdots \longrightarrow H_n(X) \longrightarrow H_n(Y) \longrightarrow H_n(C_f) \longrightarrow H_{n-1}(X) \longrightarrow \cdots$$

   (b) Let $f : S^1 \to S^1$ be the function $f(x) = x^2$. Compute the homology groups $H_*(C_f)$.

   (c) Identify the homotopy type of the space C_f for the f of the previous part. (That is, what is the "well known" space which is homotopy equivalent to C_f.)

Good Luck!