DEPARTMENT OF MATHEMATICS
University of Toronto

Topology Exam (3 hours)

Friday, September 8, 2006, 1-4 PM

The 6 questions on the other side of this page have equal value, but different parts of a question may have different weights.

Good Luck!

1. Let X and Y be metric spaces and let $f : X \to Y$ be a function.

   (a) Define "f is uniformly continuous".

   (b) Prove that if X is compact and f is continuous, then f is uniformly continuous.

2. Let X be a Hausdorff topological space.

   (a) Define "X is completely regular ($T_{3\frac{1}{2}}$)".

   (b) Prove that X is completely regular if and only if it can be embedded in a cube (a "cube" is a product space of the form $[0,1]^A$ for some set A).

   Hint and Warning. This question is a lemma often used within one of the standard constructions of the Stone-Čech compactification βX. Thus you may search your memories about βX, but of course, you cannot use βX, for logically it comes later.

3. Let X be a group with product $\star$.

   (a) Define "X is a topological group".

   (b) If $\gamma_1 : I \to X$ and $\gamma_2 : I \to X$ are paths in X, define $\gamma_1 \star \gamma_2 : I \to X$ by $(\gamma_1 \star \gamma_2)(t) = \gamma_1(t) \star \gamma_2(t)$. Show that $[\gamma_1 \star \gamma_2] = [\gamma_1][\gamma_2]$ in $\pi_1(X)$.

   (c) Show that $\pi_1(X)$ is Abelian.

4. (a) Let $p : X \to B$ be covering map and let $f : Y \to B$ be a continuous map. State in full the lifting theorem, which gives necessary and sufficient conditions for the existence and uniqueness of a lift of f to a map $\tilde{f} : Y \to X$ such that $f = p \circ \tilde{f}$.

   (b) Let $p : \mathbb{R} \to S^1$ be given by $p(t) = e^{it}$. Is it true that every map $f : \mathbb{RP}^2 \to S^1$ can be lifted to a map $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}$ such that $f = p \circ \tilde{f}$? Justify your answer.

5. A topological space X_f is obtained from a topological space X by gluing to X an n-dimensional cell e^n using a continuous gluing map $f : \partial e^n = S^{n-1} \to X$, where $n \geq 2$. Show that

   (a) $H_m(X) \cong H_m(X_f)$ for $m \neq n, n-1$.

   (b) There is an exact sequence $0 \to H_n(X) \to H_n(X_f) \to H_{n-1}(S^{n-1}) \to H_{n-1}(X) \to H_{n-1}(X_f) \to 0$.

6. (a) Let $f, g : S^n \to S^n$ be continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Show that $f \simeq a \circ g$, where a is the antipodal map.

   (b) Prove that any continuous map $f : S^{2n} \to S^{2n}$ either has a fixed point or there is a point x with $f(x) = -x$.

   (c) Prove that any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

Good Luck!