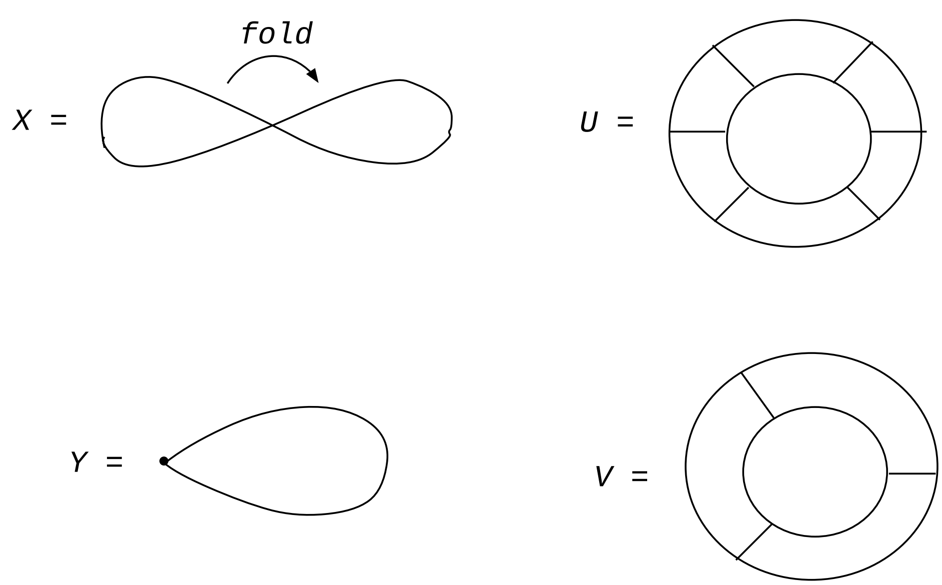
MAT 1300Y / Topology Comprehensive Exam
Monday, April 23, 2007, 9 a.m. - 12 noon

1. a) For a topological space, define the following: Hausdorff, regular, normal, compact.
   b) Show that X is Hausdorff if and only if $\Delta(X)$ is a closed subset of $X \times X$, (where $\Delta(X) = \{(x, x)\} \subset X \times X$).
   c) State Urysohn's Lemma.
   d) Show that a compact Hausdorff space is regular.

2. a) Define what it means to say that a function is a covering projection.
   b) Let X, Y, U, and V be the subsets of $\mathbb{R}^2$ pictured below. Let f: $X \to Y$ be the fold map (i.e. the map whose restriction to each circle is the identity). Let g: $U \to V$ be the map which in polar coordinates doubles the angle while leaving the radius fixed (i.e. $re^{i\theta} \mapsto re^{2i\theta}$ as complex numbers). For each of f and g, determine if the map is a covering projection and if so, decide whether or not it is a univeral covering projection.
   c) Compute the fundamental group of the space in the bottom right.



Note: The spaces include only the solid lines shown, and not the regions they surround.

3. Find the fundamental group of each of the following spaces. (Give a brief justification of your answer.) Which of these groups are abelian?
   a) S^2 with two points removed
   b) Klein bottle with a point removed
   c) Torus with two points removed

4. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of chain complexes.
   a) Describe how to construct the "connecting homomorphism" of the corresponding long exact homology sequence.
   b) Prove that the long exact homology sequence is exact at $H_n(C)$.

5. Describe one of the following two applications of Algebraic Topology.
   a) The Brouwer Fixed Point Theorem concerning functions from the n-disk D^n to itself.
   b) The existence question for vector fields on spheres.

   For the chosen application do the following.
   (i) State the theorem.
   (ii) State the relevant information from Algebraic Topology required for the proof of the Theorem. (You need not prove these statements.)
   (iii) Give the general flow of the argument including the "punch line" of the proof.

6. Let X be an n-dimensional CW-complex and let Y be a CW-complex obtained from X by attaching one more cell of dimension n. Compare the homology groups of Y with those of X. More precisely,
   a) For which values of q is it possible to have $H_q(Y) \neq H_q(X)$?
   b) For each value of q for which inequality is possible, are there restrictions on how they may differ?
   c) Give examples to illustrate the different possibilities.

7. Let X be the union $S^2 \vee S^4$ of the 2-sphere and the 4-sphere joined at a point.
   a) What is the cohomology of X, including the ring structure?
   b) Give a manifold M which has the same cohomology groups as X (although not necessarily the same ring structure).
   c) Show that X is not homotopy equivalent to a manifold and thus, in particular, X cannot be homotopy equivalent to M.