University of Toronto

Department of Mathematics

Topology Examination

Thursday, September 6, 2007, 1–4 p.m.
Duration 3 hours

1. Let X and Y be topological spaces. Let $A \subset X$ be closed and let $f : A \to Y$ be continuous. Define

$$Z_f := (X \amalg Y)/\sim$$

where $a \sim f(a)$ for all $a \in A$.

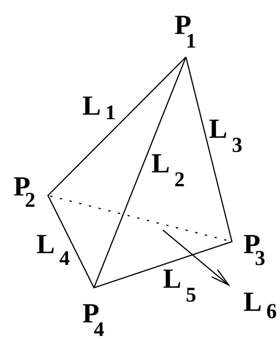
a) Define what it means for a topological space to be normal.

b) State Urysohn's lemma.

c) Show that if X and Y are normal then Z_f is normal also.

2. Let G be a path-connected topological group with identity element e. Prove that $\pi_1(G, e)$ is Abelian.

3. Let X be the outline of the tetrahedron; that is, $X = \left(\bigcup_{i=1}^{6} L_i\right) \cup \left(\bigcup_{i=1}^{4} P_i\right)$



where L_i are the edges and P_i are the vertices. Calculate $H_*(X; \mathbb{Z})$.

4. a) What does it mean to say that a manifold is orientable?

Note: There is more than one possible answer to this question. Some formulations require the existence of a smooth structure on the manifold; you may assume it is a differentiable manifold if you wish.

b) Show that real projective space $\mathbf{R}P^n$ is orientable if and only if n is odd.

5. Let T denote the (standard) 2-dimensional torus.

a) State the homology and cohomology of T including the ring structure. (Just state the results; no justification is required.)

b) State the fundamental group of T with a brief explanation of how you arrived at this answer. (Detailed proof is not required.)

c) Show that every map from the sphere S^2 to T induces the zero map on cohomology.

6. a) Let $f, g : S^n \to S^n$ be continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Show that $f \simeq a \circ g$, where a is the antipodal map.

b) Prove that any continuous map $f : S^{2n} \to S^{2n}$ either has a fixed point or there is a point x with $f(x) = -x$.

c) Prove that any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.