Part I: Complete all of the following three problems.

Problem 1.

i) State the constant rank theorem, and explain to what extent the constant rank theorem gives a local classification of smooth maps between manifolds.

ii) Let $f : M \longrightarrow N$ be a smooth map of manifolds, and let $q \in N$. Indicate whether the following are true or false, and give brief justification:

(a) If f is a submersion then $f^{-1}(q)$ is a regular submanifold of M.

(b) If q is a regular value of f, then $f^{-1}(q)$ is a regular submanifold of M.

(c) If $\operatorname{rank}(Df)$ is constant on $f^{-1}(q)$, then $f^{-1}(q)$ is a regular submanifold of M.

(d) $\operatorname{rank}(Df) = \dim N$ on $f^{-1}(q)$, then $f^{-1}(q)$ is a regular submanifold of M.

iii) Let K and L be regular submanifolds of a smooth manifold M. What does it mean for K and L to be *transverse*? Prove that if K and L are transverse, then $K \cap L$ is a regular submanifold, and compute its dimension in terms of $\dim K, \dim L, \dim M$.

Problem 2.

i) Let M be a smooth manifold covered by two open sets U, V. Let $\iota_U : U \cap V \longrightarrow U$ and $\iota_V : U \cap V \longrightarrow V$ be the inclusion maps and let $p : U \sqcup V \longrightarrow U \cup V$ be the quotient map. Write down explicitly (hint: partition of unity) the Mayer-Vietoris short exact sequence of complexes of differential forms, and prove it is exact.

ii) Using the Mayer-Vietoris long exact sequence and assuming only the Poincaré lemma, compute the de Rham cohomology groups of the spheres, S^n.

iii) Using the Mayer-Vietoris long exact sequence and assuming only the above calculations, compute the de Rham cohomology of the connected sum $\mathbb{R}P^2 \sharp \mathbb{R}P^2$, as well as $\mathbb{R}P^3$. Which of these are orientable? Why?

Problem 3.

i) For any sphere S^k, let $\iota : S^k \longrightarrow \mathbb{R}^{k+1}$ be the inclusion, and let $v_k \in \Omega^k(S^k)$ be given by

$$v_k = \iota^* \sum_{i=0}^{k} (-1)^i x^i dx^0 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^k.$$

Show that v_k is closed and that $[v_k] \neq 0$ in the top de Rham cohomology group $H^k(S^k)$.

ii) For any map $f : S^{m+n} \longrightarrow S^m \times S^n$, define $d(f) = \int_{S^{m+n}} f^*(v_m \wedge v_n)$. Does there exist a map for which $d(f) \neq 0$? Why? (Give precise argument)

iii) Show that the product of spheres $X = S^m \times S^n$ may always be embedded in $\mathbb{R}^{m+n+1}$.

iv) For any map $g : S^m \times S^n \longrightarrow S^{m+n}$, define $q(f) = \int_{S^m \times S^n} g^* v_{m+n}$. Does there exist a map for which $q(f) \neq 0$? Why? (A sketch of an argument is OK)

Part II: Complete all of the following three problems

Problem 4.

i) Let $C \subset \mathbb{R}^3$ be an embedded, unknotted circle, and let $X = \mathbb{R}^3 \backslash C$ be its complement. Compute $\pi_1(X)$ with justification.

ii) Compute the singular homology $H_\bullet(X;\mathbb{Z})$, and draw pictures of embedded submanifolds representing each of the generators of the resulting abelian groups.

iii) Is X homotopy equivalent to a compact 2-dimensional manifold? If so, which? If not, why not?

Problem 5.

i) Describe a cell complex structure on $\mathbb{R}P^n$, i.e. give the number of cells in each dimension, give the attaching maps, and give a brief sketch of why this cell complex is homeomorphic to $\mathbb{R}P^n$.

ii) Compute the homology of $\mathbb{R}P^n$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

iii) When is $\mathbb{R}P^n$ simply-connected? Why?

Problem 6.

i) Let D^2 be the closed 2-dimensional disc and $T^2 = S^1 \times S^1$, so that $Y = D^2 \times T^2$ is a manifold with boundary. Compute the relative homology groups for the pair $(Y, \partial Y)$, i.e. compute $H_\bullet(Y;\partial Y)$.

ii) Let $S^1 \subset \mathbb{R}^2$ be the usual circle; consider the resulting inclusions

$$T^2 = S^1 \times S^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 \subset S^4,$$

where the last inclusion is the one-point compactification. Let $X = S^4 \backslash T^2$, and compute $H_\bullet(X)$. [Hint: Consider the pair (S^4, X) and compare it to the first part of this question.]