Department of Mathematics
University of Toronto
Topology Comprehensive Exam
September 2009
3 hours

No aids allowed.

Problem 1.

i) Let K and L be regular submanifolds of a smooth manifold M. What does it mean for K and L to intersect *transversally*? Prove that if K and L intersect transversally, then $K \cap L$ is a regular submanifold, and compute its dimension in terms of $\dim K, \dim L, \dim M$.

ii) Let $f : M \longrightarrow N$ be a smooth map of manifolds, and let $q \in N$. Indicate whether the following are true or false, and give brief justification: [Recall: an embedded submanifold is sometimes called a "regular submanifold".]

(a) If f is a submersion then $f^{-1}(q)$ is an embedded submanifold of M.

(b) If q is a regular value of f, then $f^{-1}(q)$ is an embedded submanifold of M.

(c) If $\operatorname{rank}(f_*)$ is constant on $f^{-1}(q)$, then $f^{-1}(q)$ is an embedded submanifold of M.

(d) If $\operatorname{rank}(f_*) = \dim N$ on $f^{-1}(q)$, then $f^{-1}(q)$ is an embedded submanifold of M.

[Note: f_* denotes the derivative of f.]

Problem 2.

i) Let M be a compact orientable smooth n-manifold (without boundary) and let $\mu \in \Omega^{n-1}(M)$. Prove there exists a point $p \in M$ with $d\mu(p) = 0$.

ii) For any sphere S^k, let $\iota : S^k \longrightarrow \mathbb{R}^{k+1}$ be the usual inclusion, and let $v_k \in \Omega^k(S^k)$ be given by

$$v_k = \iota^* \sum_{i=0}^{k} (-1)^i x^i dx^0 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^k.$$

Show that v_k is closed and that $[v_k] \neq 0$ in the top de Rham cohomology group $H^k(S^k)$.

iii) Recall that $SU(2)$ is the group of complex 2×2 matrices such that $X\overline{X}^\top = 1$ and $\det X = 1$. Compute the degree of the map $S : SU(2) \longrightarrow SU(2)$ defined by $S(X) = X^4$.

Problem 3.

i) Describe the dependence of the fundamental group on the basepoint, for a connected topological space.

ii) Define the objects and morphisms in the category of chain complexes of abelian groups. Given objects A, B in this category and morphisms $f, g \in \operatorname{Hom}(A, B)$, what is a chain homotopy between f, g?

iii) Given a map $f : X \longrightarrow Y$, define the induced homomorphism $f_* : H_k(X) \longrightarrow H_k(Y)$ on singular homology.

Problem 4. Define the smooth 3-sphere, S^3, in the usual way, using two copies of $\mathbb{R}^3$ attached together via the gluing map

$$(x, y, z) \mapsto \tfrac{1}{x^2+y^2+z^2}(x, y, z).$$

Let C_1, C_2 be a pair of disjoint embedded circles in one of the copies of $\mathbb{R}^3$, given by $z = 0$ and $(x \pm 1)^2 + y^2 = \frac{1}{4}$. Let $X = S^3 \backslash (C_1 \cup C_2)$.

i) Compute the integral homology groups of X.

ii) Compute the fundamental group of X.

iii) Is X homotopy equivalent to a 2-dimensional surface? If so, which one? If not, why not?

Problem 5.

i) Describe a cell complex structure on $\mathbb{R}P^n$, i.e. give the number of cells in each dimension, give the attaching maps, and give a brief sketch of why this cell complex is homeomorphic to $\mathbb{R}P^n$.

ii) Compute the homology of $\mathbb{R}P^n$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

iii) State (but do not prove) the relationship between the fundamental group and the first homology group of a topological space.

iv) When is $\mathbb{R}P^n$ simply-connected? Why?

Problem 6.

i) State the Eilenberg-Steenrod exactness axiom for homology theories, which relates the homology groups of a pair (X, A) to the homology groups of X and A.

ii) Prove that the above exactness axiom holds for singular homology. You may use the fact that a short exact sequence of chain complexes induces a long exact sequence in homology.

iii) Let D^2 be the closed unit disk, so that $D^2 \times S^1$ is a manifold with boundary. Compute $H_*(D^2 \times S^1,\ \partial(D^2 \times S^1))$.