Name: ____________________________

Student Number: ____________________________

Midterm Exam

Math 332, Fall 2016, Geoffrey Scott

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Instructions:

- Put your name and student number on this page.
- You have 110 minutes to complete the exam. I will write the remaining time on the blackboard.
- You may ask me questions, but I cannot answer math questions or questions like "have I shown enough work?"
- I will give partial credit for some questions, so show your work.
- You may use the back of the pages as scratch paper. If you need to use the back of the pages to write your answer, make sure you write "*SEE BACK OF PAGE*" so that the grader knows where to look.
- You may leave early. Put your completed exam on the front table and have a nice evening.

Grades:

Question 1: ________ (out of 10)

Question 2: ________ (out of 10)

Question 3: ________ (out of 20)

Question 4: ________ (out of 6)

Question 5: ________ (out of 6)

Question 6: ________ (out of 10)

Total: ________ (out of 62)

Confusing Notation

Walk: A **walk** in a graph is a sequence of the form $v_0e_1v_1e_2v_2\dots e_nv_n$, where the v_i are vertices and the e_i are edges, such that the endpoints of e_i are v_{i-1} and v_i. A walk is **closed** if $v_0 = v_n$.

Trail: A **trail** is a walk in which no edge occurs more than once.

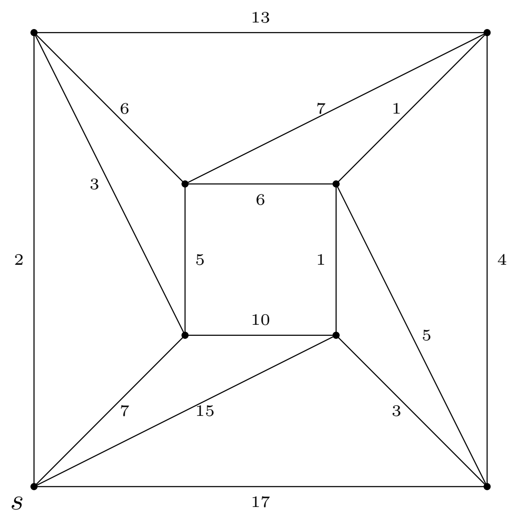
Path: A **path** is a trail in which no vertex occurs more than once, except that v_0 may equal v_n.

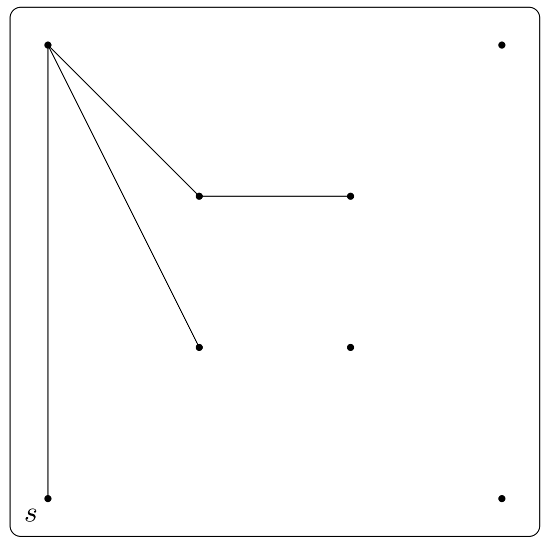
Circuit: A **circuit** is a closed trail.

Cycle: A **cycle** is a closed path.

1. Algorithms

a. Suppose you apply Dijkstra's algorithm to the graph below using the vertex labelled s as your "start" vertex. Draw the first four edges that get added to your subgraph in the box labelled ANSWER, using the vertices provided. There is a step in Dijkstra's algorithm where you might need to make an arbitrary choice between which edge to add, resulting in multiple valid answers. If this happens, any valid answer will be given full credit. (5 points)

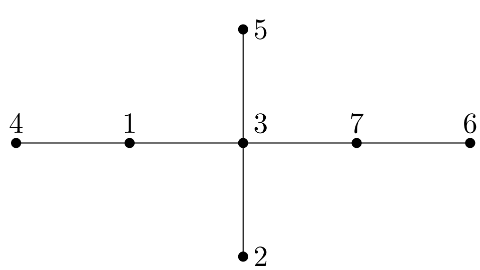




ANSWER

Scoring: 5 points total for a correct answer. 2 points if you drew the entire spanning tree instead of just the first 4 edges. 0 points for all other answers.

b. Write the Prüfer code for the tree shown below in the box labelled ANSWER. (5 points)



(3, 1, 3, 3, 7)

ANSWER

Scoring: 5 points total for a correct answer. 1 point if it is ANY other 5-digit sequence of the numbers 1 through 7. 0 points otherwise

2. Proofs from Class

a. Prove that every tree with at least one vertex has exactly $|V(G)| - 1$ edges. If you need it, you may use the fact that every tree with ≥ 2 vertices has at least two leaves, and that every tournament has a Hamiltonian path. (6 points)

Use induction on $|V(G)|$.

Base case $|V(G)| = 1$: A tree with one vertex cannot have an edge, since that edge would be a loop and G would therefore have a cycle. This means that if $|V(G)| = 1$, then $|E(G)| = 0$, as desired.

Inductive step: Assume the formula is true for all trees with n vertices where $n \geq 1$, and let $v \in V(G)$ be a leaf of a tree G with $n+1$ vertices (here, we are using the fact that $n+1 \geq 2$). Because v is a leaf, $G - v$ is still a connected and acyclic graph, hence still a tree. By the inductive hypothesis $|E(G-v)| = |V(G-v)| - 1$. Because G has one more vertex and one more edge than $G - v$, $|E(G)| = |V(G)| - 1$.

Scoring: 1 point for using induction on the size of the graph. 2 points for the base case. 3 points for the inductive step. For the base case and inductive step, points will be assigned according to the quality/correctness of that part of the proof. If you proved it some completely different way, it will be marked using the marking scheme from question 4.

b. State, but do not prove, Hall's theorem. Be precise, as though you were writing a textbook! If you the phrase "Hall's condition," state what this is. (4 points)

Let G be a bipartite graph with bipartition $V(G) = A \sqcup B$. There is a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

Scoring: 4 points total for a correct statement, 2 points total for an almost-correct statement (e.g. saying "if" instead of 'if and only if"). 0-1 points for more critical mistakes (e.g. forgetting the word "bipartite")

3. Short Answer

Give an example of the following objects, or give a short (one or two sentences should suffice) explanation why no example exists.

a. A simple connected graph with 7 vertices and 6 edges that contains an odd number of distinct cycles. For this problem, two cycles are counted as being "the same" if they use the same set of vertices. (4 points)

Does not exist: Any simple graph satisfying $|E(G)| = |V(G)| - 1$ will be a tree, and therefore, have zero cycles.
Scoring: +2 points for getting "impossible", +2 points for a good reason

b. A simple connected graph G with ≥ 3 vertices such that both G and $G - e$ have an Eulerian circuit for some $e \in E(G)$. (4 points)

Does not exist: If G has an Eulerian circuit, every vertex has even degree. But when you delete an edge, the two endpoints will have odd degree (since the graph is simple, hence loopless), so the result will have no Eulerian circuit.
Scoring: +2 points for getting "impossible", +2 points for a good reason

c. A simple connected graph G with ≥ 3 vertices such that both G and $G-e$ have a Hamiltonian cycle for some $e \in E(G)$. (4 points)

The graph K_4 works.
Scoring: 4 points for a correct example, no partial credit

d. A simple graph with at least two vertices and degree sequence $(|V(G)|-1, |V(G)|-2, \ldots 3, 2, 1, 0)$. (4 points)

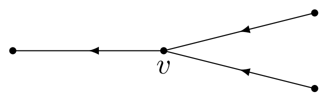
Does not exist: Because G is simple, any vertex with degree $|V(G)| - 1$ must be adjacent to every other vertex in the graph. But there is a vertex of degree 0, which it cannot be adjacent to.
Scoring: +2 points for getting "impossible", +2 points for a good reason

e. A simple connected graph G such that the maximum size of a matching in G is 2, and the minimum size of a vertex cover in G is 3. (4 points)

The cycle graph C_5 works.
Scoring: 4 points for a correct example, no partial credit

4. New Proof

In a directed graph, the **indegree** of a vertex v, written $d_i(v)$ is the number of edges incident to v that are directed *towards* v, and the **outdegree**, written $d_o(v)$ is the number of edges incident to v directed *away from* v. For example, the vertex v in the graph below has indegree 2 and outdegree 1.



Prove that for any directed graph G,

$$\sum_{v \in V(G)} d_i(v) = \sum_{v \in V(G)} d_o(v).$$

(6 points)

Every edge contributes 1 to the sum $\sum_{v \in V(G)} d_i(v)$, and contributes 1 to the sum $\sum_{v \in V(G)} d_o(v)$. Therefore,

$$\sum_{v \in V(G)} d_i(v) = |E(G)| = \sum_{v \in V(G)} d_o(v).$$

Scoring: roughly 6 points for a correct proof. 4-5 points for an almost-correct proof with minor mistakes. 2-3 points for a solution that demonstrates at least some correct thought, but makes no/minimal progress towards proving the statement.

5. New Proof 2

Let e be an edge of K_n. Calculate the number of spanning trees of $K_n - e$. Explain how you got your answer. (6 points)

Let k denote the number of spanning trees of K_n containing the edge e. Notice the following observations:

- There are $n^{(n-2)}$ spanning trees total, each of which contains $n-1$ edges.
- There are $n(n-1)/2$ edges in K_n. Each of these edges is contained in k different spanning trees.

Now consider the disjoint union of all spanning trees of K_n. We have two ways to count all the edges in this gigantic graph. By the first observation: $(n-1)n^{n-2}$. By the second observation: $kn(n-1)/2$. Setting these equal gives $kn/2 = n^{n-2}$, so $k = 2n^{n-3}$. This means there are $n^{n-2} - 2n^{n-3}$ spanning trees of K_n which do *not* contain e, or $\mathrm{st}(K_n - e) = (n-2)n^{n-3}$.

Note: It is very tempting to try to prove this using the edge deletion-contraction method. In fact, that's the first approach I tried. I'll be very generous with grading and give a lot of partial credit to any reasonable attempt at solving the problem.

Scoring: 6 points if you got the correct formula, no questions asked. 5 points if you attempted deletion/contraction and got stuck. 4 points if you attempted delection/contraction and mistakenly assumed that contracting K_n by an edge gives K_{n-1} to get a wrong answer. All other methods of proof will be given 5 points if significant progress/effort was made, but you got stuck, and 4 points if significant progress/effort was made, and you said some false statement to arrive at an incorrect answer.

6. New Proof 3

Consider a deck of 52 cards where each card has an integer from 1 to 13 written on it, and each such number is written on exactly four cards[1]. Suppose I shuffle the cards and deal them face-down into 13 piles, each pile containing four cards. Prove that it is possible for me to examine each pile, then pick exactly one card from each pile so that in total I've picked exactly one card of each number. (10 points)

Let G be the graph with vertices $\{N_1, \ldots, N_{13}\}$ corresponding to the numbers $1, \ldots, 13$, and vertices $P_1, \ldots, P_{13}$ corresponding to the piles $1, \ldots, 13$. Vertex N_i is connected to vertex P_j if pile P_j contains a card with number N_i. This is a bipartite graph, and a way of selecting the cards so that I've picked one card of each number corresponds to finding a perfct matching. Therefore, to complete the proof it suffices to verify Hall's condition for the set $A = \{P_1, \ldots, P_{13}\}$. Any subset $S \subseteq A$ of piles contains $4|S|$ cards. Because there are only four cards per number, there must be at least $4|S|/4 = |S|$ different numbers present in the set of cards in the piles S. Therefore, $|N(S)| \geq |S|$.

Note: There is another way to do this problem where you set up the graph so that it has an edge for each card (i.e. so if a pile has three 2's and one 7, then the corresponding vertex would have three edges to the vertex N_2 and one edge to vertex P_7). You can show that this bipartite graph is 4-regular, so has a perfect matching by the result from class that states that every regular bipartite graph has a perfect matching.

Scoring: +4 points if you managed to set up some bipartite graph for which a perfect matching would result in a method of choosing cards. +6 points if you successfully proved that the bipartite graph has a perfect matching. Points will be deducted from each of these categories for minor errors

[1] This is just a "standard" deck of cards without jokers. The only purpose of this sentence is in case anyone is unfamiliar with what a "standard" deck of cards is