MAT332 - Fall 2016 - Homework 1

Due Tues. Sept 27

If I ask you to *draw* or *show* an example of something, no explanation is required. If I ask you to *explain* or *justify* or *prove* something, you need to thoroughly explain why your answer is correct.

1. For each of the following, draw an example of the object described or explain why such an object cannot exist.

a. A simple graph with degree sequence $(6, 4, 3, 2, 2, 1, 1, 1, 0)$.

b. A simple graph with degree sequence $(8, 4, 0)$.

c. A simple graph where no two vertices have the same degree.

d. A 1-regular graph on 4 vertices.

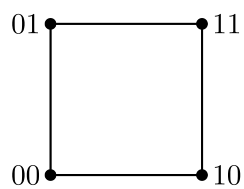
e. A simple graph that has an Eulerian circuit but no Hamiltonian cycle.

f. A simple graph that has a Hamiltonian cycle but no Eulerian circuit.

g. A simple graph that has a Hamiltonian cycle *and* an Eulerian circuit *and* at least one vertex of degree ≥ 3.

h. A simple graph with an Eulerian circuit that has an odd number of edges.

2. Let H_n be the graph which has a vertex for every possible n-digit binary sequence (i.e. a string of 0's and 1's), and two vertices are connected if the two corresponding binary sequences differ in exactly one of the digits. A picture of H_2 is shown below.



a. How many edges and vertices are there in H_n? Explain your answer.

b. What is the degree sequence of H_n? Explain your answer.

c. Prove that H_n has a Hamiltonian cycle for all $n \geq 2$. (hint: induction).

3. Prove that every simple graph with 10 vertices and 28 edges contains a cycle of length 4.

4. Let n be any even positive integer, and let G be a $2n$-regular simple graph. Prove that it is possible to color the edges of G red and blue in such a way that every vertex is incident to n red edges and n blue edges.

5. You are at a river with an empty 3 liter bucket (bucket A) and an empty 5 liter bucket (bucket B). There are three operations you can perform.

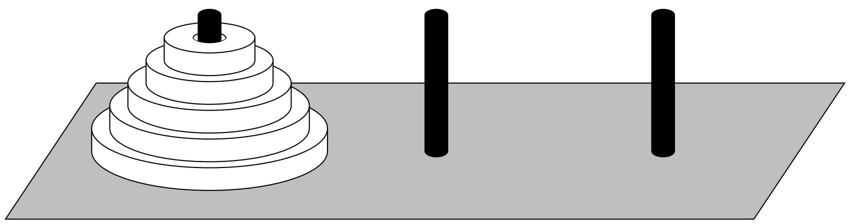
- You may fill a bucket that is not already full from the river.
- You may empty a non-empty bucket into the river.
- You may transfer water from one non-empty bucket to another bucket. If there is not enough room to transfer *all* the water, you transfer all the water that is possible, and leave the remaining water in the original jug.

A popular children's puzzle asks you to find a way to measure exactly four liters using these jugs, and the popular way to solve this puzzle is to guess and check. We will represent this puzzle using graph theory. Call a pair of positive integers (a, b) *attainable* if it is possible to perform a series of operations so that bucket A contains exactly a liters of water and bucket b contains exactly b liters of water.

a. Draw a directed graph that has one vertex for each attainable pair of integers, and has a directed edge from the vertex representing (a, b) to (a', b') if it is possible to go from the configuration represented by (a, b) to the configuration represented by (a', b') using just one of the above operations.

b. Write down a walk in your graph that represents a solution to the puzzle.

6. The *Tower of Hanoi* is a puzzle consisting of n discs with differing radii and a board with three pegs. The initial configuration of the puzzle for $n = 5$ is shown below.



The goal of the puzzle is to move the discs so that they are all on the rightmost peg. A legal move consists of moving the top disc from any of the three pegs to any different peg, as long as this does not cause a larger disc to be above a smaller disc on the same peg. Call a configuration of discs *legal* if no larger disc is above a smaller disc on the same peg. Let Han_n be the graph whose vertices are legal configurations of the n-disc tower of Hanoi, and two vertices are connected if the corresponding legal configurations differ by a single move. (unlike in the bucket puzzle, every move is reversible, so we do not need to direct the edges of this graph)

a. Prove (using induction) that Han_n is connected. Explain in words what this means in the context of the puzzle.

b. How many vertices does Han_n have? You do not need to justify your answer.

c. For which positive integers n does Han_n have an Eulerian circuit? Justify your answer.

7. Suppose G is a graph with an Eulerian circuit, and $e, f \in E(G)$ share an endpoint. Prove that G has an Eulerian circuit where e, f appear as consecutive edges, or give an example for which this isn't true.

8. Let G be a connected graph, and let n be the length of the longest trail in G. Prove that if two trails in G have length n, then there must be some vertex $v \in V(G)$ that is contained in both trails.