Algebra Notes

Oct. 26: Constructible Numbers

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We're finally ready for our second application of ring theory (the first was the Chinese remainder theorem). This week, we'll show how the theory of field extensions helps us solve basic questions about geometry that stumped the Greeks. Suppose you have a collection of points, lines, and circles on the plane $\mathbb{R}^2$, and you have a compass and straightedge. There are three basic operations you can perform using these tools.

Compass and Straightedge Operations

1. Draw an (infinite) straight line between two points.
2. Set your compass radius to the distance between two points, and draw a circle around a point. This third point may be the same as or may be different from your original two.
3. Draw a point at the intersection of two circles, of two lines, or of a circle and a line.

A natural question to ask is:

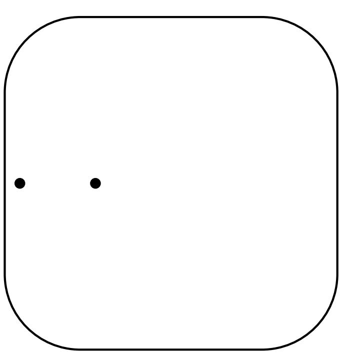
> What complicated operations can I perform as a sequence of these basic operations?

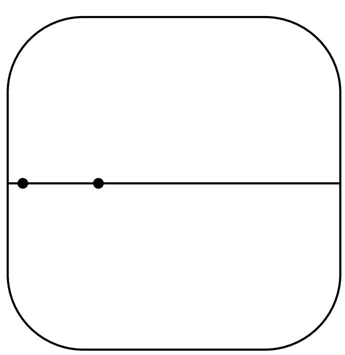
For example, if you have two points that are a certain distance d apart, can you construct a equilateral triangle with side lengths d? What about a square, pentagon, or hexagon? Can you bisect angles? Trisect angles? Can you construct parallel lines? Perpendicular lines?

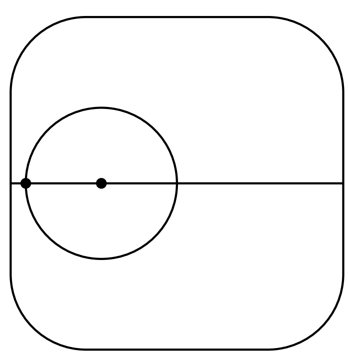
Constructibility

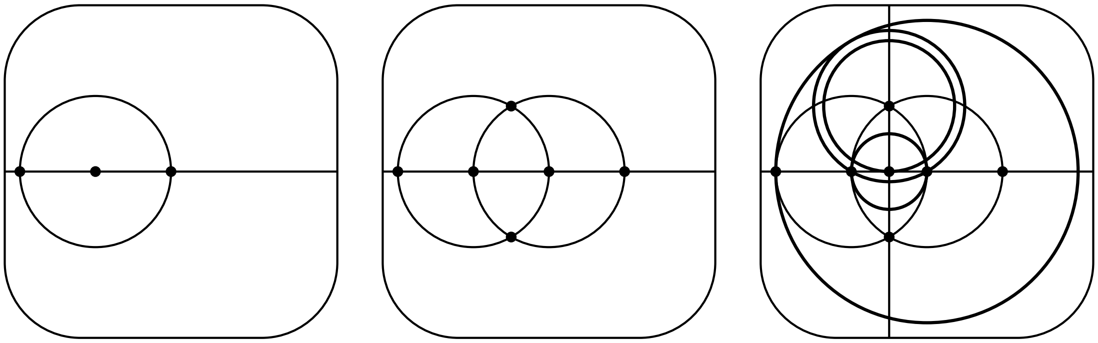
Definition: Start with two points on the plane $\mathbb{R}^2$ that are distance 1 apart. A point, line, or circle on $\mathbb{R}^2$ is **constructible** if can be drawn in a finite number of steps using the operations above. An angle θ is **constructible** if there exists two constructible lines that intersect at an angle θ. A real number k is **constructible** if there are two constructible points that are distance $|k|$ apart.

Let's get a feel for constructible lines, points, circle, and numbers by seeing what we can draw just by starting with two points of distance 1 apart.









Clearly, you can get a lot of lines and circles and points just from these simple operations. Let's be more systematic in our study of constructible numbers. What properties do they have?

Claim: If a and b are constructible numbers, then so is $a + b$.

Proof: If either of a or b is zero, the claim is trivial. Otherwise, draw a line through the two points p, q that are distance $|a|$ apart, then a circle of length $|b|$ around q. One of the points where this circle intersects the line will be a distance of $|a + b|$ from p.

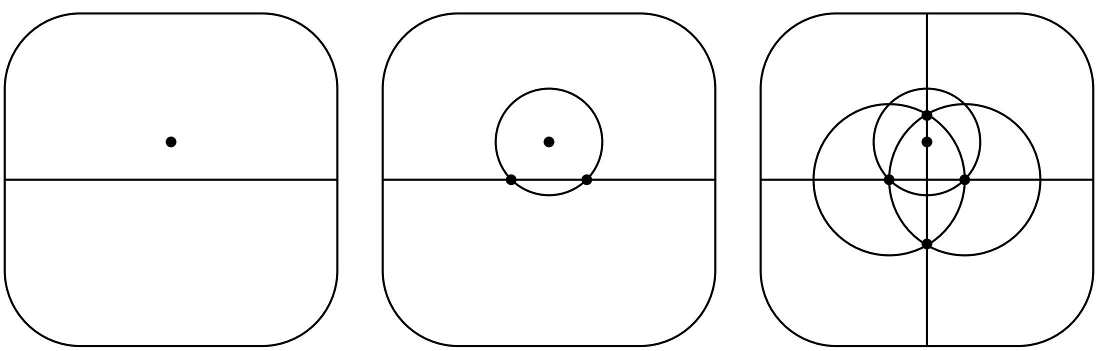
Claim: If a and b are constructible numbers, so is $a - b$

Proof: If b is constructible, then so is $-b$. So $a + (-b)$ is constructible.

As a consequence of the two claims above, every integer is constructible. We also want to show that ab and a/b are constructible, but first we show two general tricks for constructions.

Claim: If p is a constructible point and L a constructible line, then the line through p perpendicular to L is also constructible.

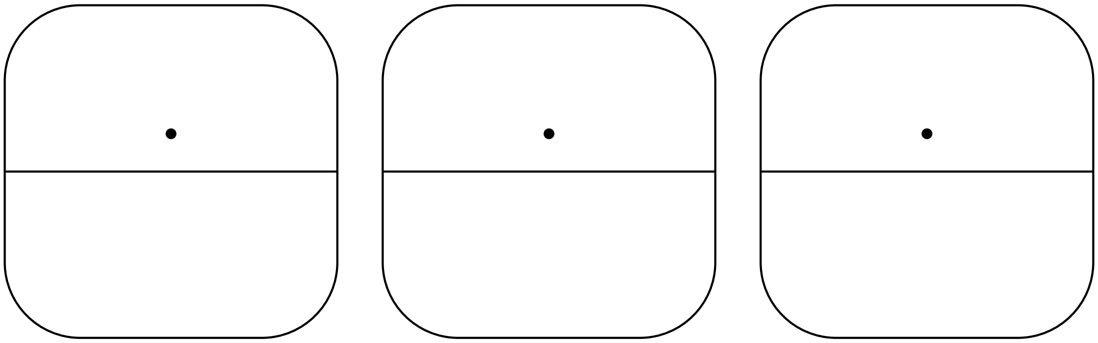
Proof:



Note that the above proof works even when p is on the line L.

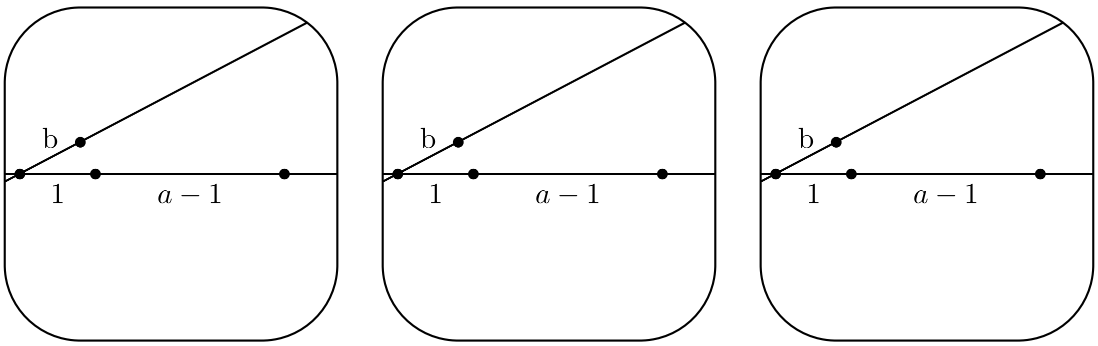
Claim: If p is a constructible point and L a constructible line, then the line through p parallel to L is also constructible.

Proof: (fill me in!)



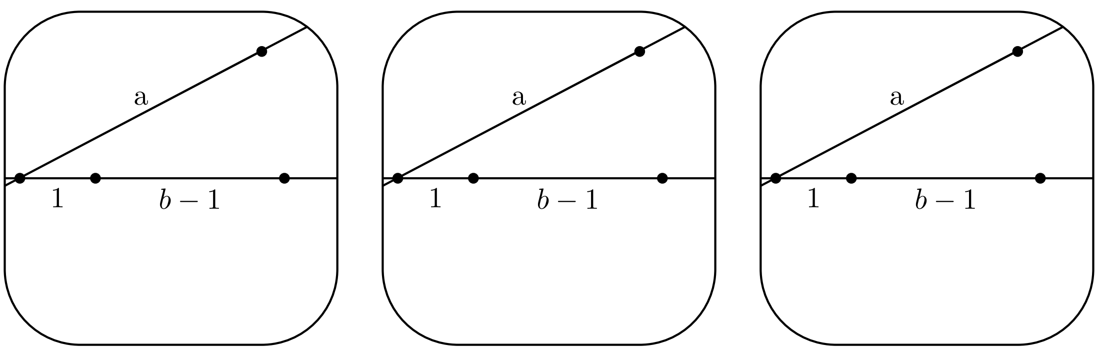
Claim: If $a, b \in \mathbb{R}$ are constructible, then so is ab.

Proof: (fill me in!)



Claim: If $a, b \in \mathbb{R}$ are constructible, then so is a/b.

Proof: (fill me in!)



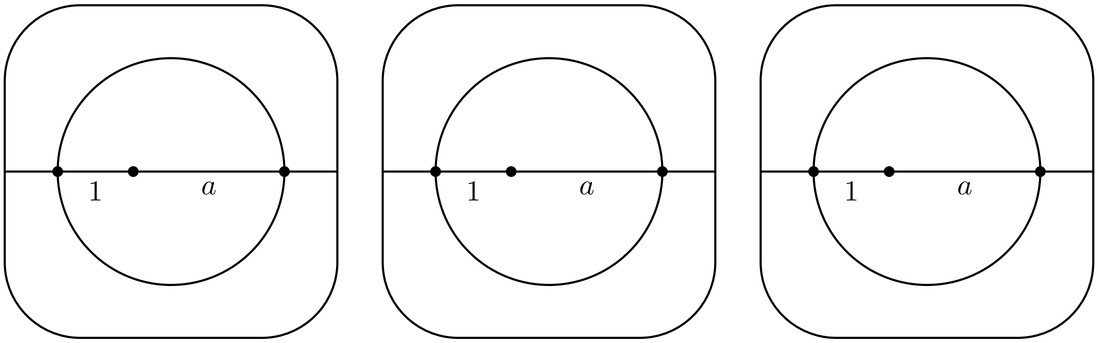
Theorem: The set of constructible numbers is a subfield of $\mathbb{R}$.

Proof: We just need to show that it is a subring in which every nonzero element has an inverse. The fact that it is a subring follows from the fact that it is closed under addition, subtraction, and multiplication. The fact that every nonzero element is a unit follows from the fact that 1 is constructible, and that $1/a$ is constructible if a is constructible.

We already know that there are some irrational constructible numbers (for example $\sqrt{2}$). The next two give some other examples of constructible numbers.

Claim: If $a > 0$ is constructible, so is $\sqrt{a}$.

Proof: (fill me in!)



Claim: Let θ be any angle (even one which is not already known to be constructible). If the number $\cos(\theta)$ is constructible, then so is the number $\sin(\theta)$.

Proof: (fill me in!)