Most of the problems below are necessary to complete to understand the minicourse. Problems marked with $*$ are not. Do not attempt the problems marked $*$ until you have completed all the others.

Problems for after the first lecture

If you aren't familiar with the Hodge star operator, you can use the following description of it: on Minkowski space $\mathbb{R}_t \times \mathbb{R}^3_{x,y,z}$ with metric $-dt^2 + dx^2 + dy^2 + dz^2$, the Hodge star operator is a $C^\infty(M)$-linear map

$$\star : \wedge^p T^*M \to \wedge^{4-p} T^*M$$

given by

$$\begin{aligned} dt &\mapsto (-1) dx \wedge dy \wedge dz \\ dx &\mapsto (-1) dt \wedge dy \wedge dz \\ dy &\mapsto (-1) dt \wedge dz \wedge dx \\ dz &\mapsto (-1) dt \wedge dx \wedge dy \end{aligned}$$

$$\begin{aligned} dt \wedge dx &\mapsto (-1) dy \wedge dz & \qquad dx \wedge dy &\mapsto dt \wedge dz \\ dt \wedge dy &\mapsto (-1) dz \wedge dx & \qquad dy \wedge dz &\mapsto dt \wedge dx \\ dt \wedge dz &\mapsto (-1) dx \wedge dy & \qquad dz \wedge dx &\mapsto dt \wedge dy \end{aligned}$$

$$\begin{aligned} dx \wedge dy \wedge dz &\mapsto (-1) dt \\ dt \wedge dy \wedge dz &\mapsto (-1) dx \\ dt \wedge dz \wedge dx &\mapsto (-1) dy \\ dt \wedge dx \wedge dy &\mapsto (-1) dz \end{aligned}$$

Warm up (short answer). Find the mistake in the following argument.

> Suppose you have a electromagnetic tensor F on some contractible subset V of Minkowski space, and there is some smaller open set $U \subseteq V$ on which F vanishes identically. Then, because we can pick any $A \in \Omega^1(V)$ to be our electromagnetic potential (since V is contractible), and F vanishes on U, we can choose an A that also vanishes on U. Then, if we perform experiments on particles that are confined to stay in U (and disregard all "quantum tunneling" effects that could bring a particle out of U), the $\int qA$ term in the action of the particle vanishes, so there is no electromagnetic influence on the particles.

1. Let M be Minkowski space. Suppose you've chosen a spacetime splitting for M with coordinates (t, x, y, z). Let $E(t, x, y, z)$ and $B(t, x, y, z)$ be vector-valued functions in these coordinates which represent the electric and magnetic fields in the presence of some electric density ρ and current j. Let $F \in \Omega^2(M)$ be the corresponding electromagnetic 4-tensor and J the corresponding electromagnetic current 1-form. Verify that Maxwell's equations are equivalent to

$$\begin{aligned} dF &= 0 \\ \star d \star F &= J \end{aligned}$$

2. **(*)** Let E and B be as in problem 1. Assume they satisfy Maxwell's equations. Consider a change of coordinates for M obtained by a Lorentz boost in the x-direction of velocity v. Find expressions for the electric and magnetic fields in these new coordinates. (*)

3. Let $\hat{r}$ denote the vector field on $\mathbb{R}^3$ consisting of vectors of unit length that point radially away from the origin. Recall from grade school that the electric field caused by an electric charge of charge q at the origin is given by

$$\frac{q\hat{r}}{4\pi r^2}.$$

- Write down the electromagnetic tensor for a point charge which is stationary at the origin. Your solution should have a singularity along the line $L = \{x = y = z = 0\}$ in M.
- Viewing F as a 2-form on $M \backslash L$, verify that F satisfies Maxwell's equations with $J = 0$.

4. Suppose $F \in \Omega^2(M)$ satisfies Maxwell's equations for $J = 0$.

- Show that $\star F$ also satisfies Maxwell's equations.
- Suppose you have chosen a spacetime splitting, so that F can be expressed in terms of electric and magnetic fields. Find explicit formulas for the electric and magnetic fields of $\star F$ in terms of the electric and magnetic fields corresponding to F.

5. Let F_{em} be the electromagnetic tensor described in question 4, and let $F_{mm} = \star F$. (the subscripts *em* and *mm* stand for "electric monopole" and "magnetic monopole", respectively). Write an expression in coordinates for F_{mm} as a 2-form, and also write an expression in coordinates for the electric and magnetic fields correspending to F_{mm} in the same spacetime splitting used in problem 4.

6. Let A_+ and A_- be the 1-forms on $U_+ = \mathbb{R}^3 \backslash \{x = y = 0, x \leq 0\}$ and $\mathbb{R}^3 \backslash \{x = 0 = 0, z \geq 0\}$, respectively, given by

$$A_+ = \frac{YdZ - ZdY}{4\pi R(X + R)} \qquad\qquad A_- = \frac{ZdY - YdZ}{4\pi R(R - X)}$$

- If you want to practice your exterior differentiation, verify that $dA_+ = F_{mm}|_{U_+}$ and that $dA_- = F_{mm}|_{U_-}$. Otherwise, just accept that it's true.
- Let $\gamma_\epsilon(t) : [0, 2\pi] \to M$ be the parametrized circle

$$X = 1 \qquad Y = \epsilon\cos(t) \qquad Z = \epsilon\sin(t)$$

  Calculate $\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} A_+$ and $\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} A_-$. You can do this directly or by using Stokes' theorem.
- According to the path integral formulation of quantum mechanics, if A satisfies $dA = F$ in a contractible domain, then the integral $\int_\gamma A$ describes the amount of phase that a particle of charge 1 "accumulates" as it travels along γ. You got different answers for this number depending on whether you used A_- or A_+ above. Explain the paradox. In reality, how much phase *would* a particle of charge 1 accumulate when traveling around γ_ϵ for ϵ small?

Problems for after the third lecture

Warm up (short answer).

- What is the difference between the definition of a circle bundle and a principal $U(1)$ bundle? Does a circle bundle always admit a structure of a principal $U(1)$ bundle? If so, how many different ways are there to make a circle bundle into a principal $U(1)$ bundle? Is a principal $U(1)$ bundle always a circle bundle?
- Prove or find a counterexample: A principal $U(1)$ bundle is trivial (i.e. equivariantly isomorphic to $M \times U(1)$ with the product action) iff it has a global section. Prove or find a counterexample: A vector bundle is trivial iff it has a global section.

1. (Do this problem only if you would like more pratice with hands-on coordinate calculations of connections and their curvature. If not, read through and do the next) Consider the principal $U(1)$ bundle

$$\pi : \mathbb{R}^4 \backslash \{0\} \to \mathbb{R}^3 \backslash \{0\}$$
$$x \mapsto x i \overline{x}$$

where the expression for π uses the identification $\mathbb{R}^4 \cong \mathbb{H}$ and $\mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$. The $U(1)$ action is given by $x \cdot e^{i\theta} = xe^{i\theta}$.

- Verify that $\alpha = r^{-2}(-x_1 dx_0 + x_0 dx_1 + x_3 dx_2 - x_2 dx_3)$ on $\mathbb{R}^4 \backslash \{0\}$ is a connection form.
- Verify that

$$s_+ : U_+ \to \mathbb{R}^4$$
$$Xi + Yj + Zk \mapsto \frac{1}{\sqrt{2(X+R)}}((R+X) + 0i - Zj + Yk)$$
$$s_- : U_- \to \mathbb{R}^4$$
$$Xi + Yj + Zk \mapsto \frac{1}{\sqrt{2(R-X)}}(Y + Zi + 0j + (R-X)k)$$

  are sections of π over the sets U_+ and U_-.
- Verify that

$$s_+^*(\alpha) = \frac{YdZ - ZdY}{2R(X+R)} \qquad \text{and that} \qquad s_-^*(\alpha) = \frac{ZdY - YdZ}{2R(R-X)}.$$

- Let $F_{mm} \in \Omega^2(\mathbb{R}^3 \backslash \{0\})$ be the form

$$F_{mm} = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{4\pi r^3}.$$

- Verify that $\frac{1}{2\pi} d(s_+^* \alpha) = F_{mm}|_{U_+}$ and that $\frac{1}{2\pi} d(s_-^* \alpha) = F_{mm}|_{U_-}$.

2. Let F_{mm} be the same form from the last problem, and let T_F be the corresponding current on all of $\mathbb{R}^3$. Show that

$$dT_F = T_{\{0\}}$$

3. (*) Consider the electromagnetic tensor F due to a charged particle that you calculated in problem 3 of the last problem set, and let T_F be the corresponding current. Calculate the current $T_J = \star d \star T_F$ by describing how the current T_J acts on compactly supported forms. (here, T_J is a "current" in the mathematical sense and also a "current" in the common sense of electromagnetism).

4. Let $x_1, \ldots, x_N$ be distinct points in a connected compact oriented manifold M, and $a_1, \ldots, a_N \in \mathbb{R}$.

- Describe necessary and sufficient conditions for the current $\sum a_i T_{\{x_i\}}$ to be closed.
- Describe necessary and sufficient conditions for the current $\sum a_i T_{\{x_i\}}$ to be exact.

5. In lecture, we reviewed that a connection on a $U(1)$ bundle has a curvature form ω for which $[\omega/2\pi]$ is an integral class. Correspondingly for gerbes, every 1-connection on a gerbe has a Dixmier-Douady form which represents an integral cohomology class. Reverse this construction to prove the following:

- Let $F \in \Omega^2(M)$ be a closed 2-form for which $[F]$ is an integral class. Prove that there is a principal S^1-bundle with connection, whose curvature form is exactly F (not merely "in the same cohomology class as F").
- Let $F \in \Omega^3(M)$ be a closed 3-form for which $[F]$ is an integral class. Prove that there is a gerbe with 1-connection whose Dixmier-Douady 3-form is exactly F.