Sept 20: Hamiltonian Cycles and Tournaments

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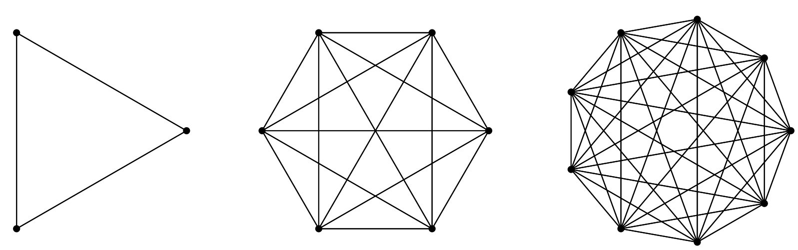
Last week, we studied walks that use no edge more than once (trails) and walks that use each edge exactly once (Eulerian trails). Today, we study trails that use no vertex more than once (paths) and trails that use each vertex exactly once (Hamiltonian paths).

> **Definition:** A **path** is a trail that uses each vertex at most once, (except the first vertex may equal the last vertex). A **cycle** is a closed path. A path or cycle is **Hamiltonian** if it uses every vertex.

Recall that a train enthusiast might plan a vacation by finding an Eulerian circuit in the graph of train routes so that she could ride on all the train tracks and end where she started. A train *station* enthusiast would want to find a Hamiltonian cycle. If you represent your social friend network as a graph (where vertices represent people and edges represent friendship), then a Hamiltonian cycle represents a way to seat your friends around a circular dinner table so that everyone is friends with the two people sitting next to them.

In general, Hamiltonian paths and cycles are *much* harder to find than Eulerian trails and circuits. We will see one kind of graph (complete graphs) where it is always possible to find Hamiltonian cycles, then prove two results about Hamiltonian cycles.

> **Definition:** The **complete graph on n vertices**, written K_n, is the graph that has n vertices and each vertex is connected to every other vertex by an edge.



K_3 K_6 K_9

Remark: For every $n \geq 3$, the graph K_n has $n!$ Hamiltonian cycles: there are n choices for where to begin, then $(n-1)$ choices for which vertex to visit next, then $(n-2)$ choices for which vertex to visit after that, and so on. Because the graph is complete, there will always be an edge that will take you to the next vertex on your list. After the final vertex, take the edge that connects back to your starting vertex.[1]

In general, having more edges in a graph makes it more likely that there's a Hamiltonian cycle. The next theorem says that if all vertices in a graph are connected to *at least half* of the other vertices, there is guaranteed to be a Hamiltonian cycle.

[1] Ask yourself: where did we use the condition that $n \geq 3$? Try and see what happens if $n = 2$.

Theorem: Let G be a simple graph with at least 3 vertices. If every vertex of G has degree $\geq |V(G)|/2$, then G has a Hamiltonian cycle.

Proof: Assume that G satisisfies the condition, but does *not* have a Hamiltonian cycle. If it is possible to add edges to G so that the result still a simple graph with no Hamiltonian cycle, do so. Continue adding edges until it becomes impossible to add edges without creating a cycle. Call this new graph G'.

Because G' has no Hamiltonian cycle and has ≥ 3 vertices, it cannot be a complete graph – i.e. there are vertices $v, w \in V(G')$ that are *not* connected by an edge. Adding the edge vw to G' will result in a graph having a Hamiltonian cycle; deleting the edge vw from this cycle produces a Hamiltonian path in G' from v to w. Let $(v, v_2, v_3, \ldots, v_{n-1}, w)$ be the vertices in this path in order (so $|V(G')| = n$).

Define two subsets of the set $\{2, 3, \ldots, n-2\}$ as follows

$$A = \text{all numbers } i \in \{2, 3, \ldots, n-2\} \text{ such that } vv_{i+1} \text{ is an edge of } G'$$
$$B = \text{all numbers } i \in \{2, 3, \ldots, n-2\} \text{ such that } v_i w \text{ is an edge of } G'$$

Notice that every edge with endpoint v is accounted for in the set A except for the edge vv_2. Because $d(v) \geq n/2$, this means $|A| \geq n/2 - 1$. Similarly, $|B| \geq n/2 - 1$. Because the set $\{2, 3, \ldots, n-2\}$ has $n-3$ elements in it, and $|A| + |B| \geq n-2$, at least one element of $\{2, 3, \ldots, n-2\}$ is in $A \cap B$. Then there is a Hamiltonian cycle with vertices

$$(v, v_2, v_3, \ldots, v_i, w, v_{n-1}, v_{n-2}, \ldots, v_{i+1}, v)$$

which gives the desired contradiction.

Tournaments

Last class, we learned the definition of a directed graph (*digraph* for short): a graph where each edge has a prescribed direction. The definitions of a *walk, trail, Eulerian circuit, path, cycle, Hamiltonian cycle* generalize to the context of digraphs by insisting that edges may only be traversed in this prescribed direction, much like a one-way street. We learned that complete graphs with at least 3 vertices always have Hamiltonian cycles. What if the edges are oriented? Oriented digraphs whose underlying graph is complete are called *tournaments*.

Definition: A **tournament** on n vertices is a directed graph whose underlying graph is K_n.

Theorem: Every tournament has a Hamiltonian path (*not* necessarily a cycle!).

Proof: We prove this by induction on the number of vertices. If a tournament has just one vertex, the claim is true – the path containing just the single vertex is Hamiltonian. Now assume we know the claim is true for all tournaments on n vertices, and consider a tournament G on $n+1$ vertices. Let v be any vertex in G. If we delete v (and all edges with v as an endpoint), the remaining tournament on n edges must have a Hamiltonian path by the inductive hypothesis. Label the vertices in this path $v_1, v_2, \ldots v_n$. In the original tournament G, consider the possible orientations of the edges incident to v:

There are three cases:

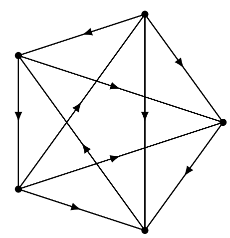


Figure 1: A tournament with underlying graph K_5

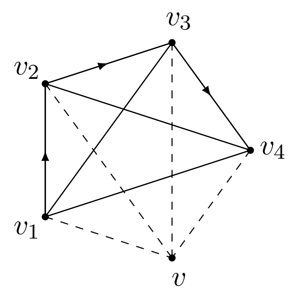
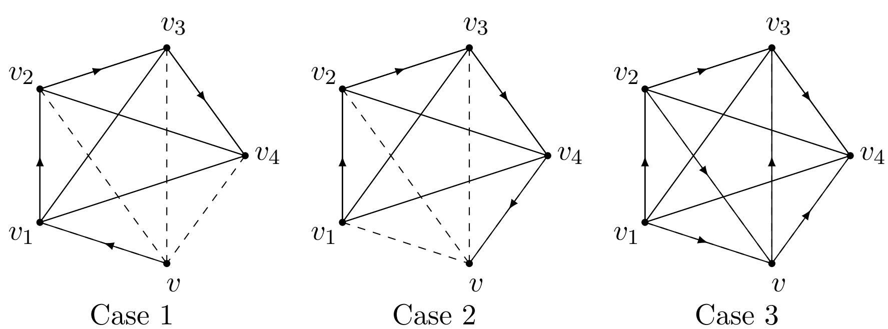


Figure 2: Consider the possible orientations of the dashed lines

Case 1: If vv_1 is an edge (i.e. the edge containing v and v_1 is oriented towards v_1), then there is a Hamiltonian path with vertex order $v, v_1, \ldots, v_n$.

Case 2: If $v_n v$ is an edge, there is a Hamiltonian path with vertex order $v_1, \ldots, v_n, v$.

Case 3: If Case 1 and Case 2 do not hold, as you look through the edges incident to v in order (starting with the edge containing v_1, then the edge containing v_2, etc...) there must come a point where the edges switch from pointing *towards* v to pointing *away* from v. That is, there is at least one number $1 \leq i \leq n-1$ for which $v_i v$ is an edge and vv_{i+1} is an edge. Then there is a Hamiltonian path with vertex order $v_1, v_2, \ldots, v_i, v, v_{i+1}, \ldots, v_n$.



Graph Isomorphism

When are two graphs "the same"? According to the definition of a graph, the two sets of vertices must be the *exact same set*, the set of edges must be the *exact same set*, and the function from edges to vertices must be the *exact same function*. But if you want to study just the *structure* of how the vertices are connected to each other, rather than the details of exactly what the vertices represent, then this notion of "sameness" is too restrictive, and the notion of *isomorphic* graphs is more appropriate.

Definition: An **isomorphism** between graphs G and H is a pair of bijections $\varphi : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that for every $e \in E(G)$, if e has endpoints v, w then $\phi(e)$ has endpoints $\varphi(v)$ and $\varphi(w)$.

If G and H are simple graphs, then an isomorphism can be described just using a bijection $\varphi : V(G) \to V(H)$ having the property that two vertices $v, w \in V(G)$ are adjacent in G if and only if $\varphi(v)$ and $\varphi(w)$ are adjacent in H.

Two graphs G, H are said to be **isomorphic** if there is an isomorphism from G to H.