Supplementary Questions for HP Chapter 10

1. Evaluate $\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$, using the identity $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$.

2. Show by example that

(a) $\lim_{x\to a}[f(x)+g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

(b) $\lim_{x\to a}[f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

3. Evaluate

(a) $\lim_{x\to 0} \frac{\sqrt[3]{1+cx}-1}{x}$

(b) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{\sqrt{x}-1}$

4. A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x in the domain of f, and an odd function if $f(-x) = -f(x)$ for all x in the domain of f.

Suppose $f(x)$ is even and $\lim_{x\to a^+} f(x) = L$.

(a) Find, if possible, $\lim_{x\to -a} f(x)$.

(b) Find, if possible, $\lim_{x\to -a^-} f(x)$.

(c) Find, if possible, $\lim_{x\to -a^+} f(x)$.

(d) Repeat (a) to (c) for an odd function $f(x)$.

5. Let $F(x) = \frac{x^2-1}{|x-1|}$

(a) Find

i) $\lim_{x\to 1^+} F(x)$ ii) $\lim_{x\to 1^-} F(x)$

(b) Does $\lim_{x\to 1} F(x)$ exist?

(c) Sketch the graph of F.

6. **(a)** Let r be any positive number. Show that $\frac{n}{r}\ln\left(1+\frac{r}{n}\right)$ is the slope of the straight line connecting $g(1)$ and $g(1+\frac{r}{n})$ for the function $g(x) = \ln(x)$.

(b) In light of question (a), what is $\lim_{n\to\infty} \frac{n}{r}\ln(1+\frac{r}{n})$ in terms of the function g? It may help to write $h = \frac{r}{n}$.

(c) Write $(1+\frac{r}{n})^n = e^{n\ln(1+\frac{r}{n})}$. Show $\lim_{n\to\infty}(1+\frac{r}{n})^n = e^r$, remembering $g'(1) = 1$.

7. **(a)** In terms of compound interest, explain why it is reasonable to expect that

$$\left(1+\frac{r}{n+1}\right)^{n+1} > \left(1+\frac{r}{n}\right)^n$$

where $r > 0$, n is a positive integer.

(b) Show that $\left(1+\frac{r}{n+1}\right)^{n+1} > \left(1+\frac{r}{n}\right)^{n}$ using the identity $a^b = e^{b\ln a}$ and using the fact that $\frac{n}{r}\ln\left(1+\frac{r}{n}\right)$ is the slope of the straight line connecting $g(1)$ and $g(1+\frac{r}{n})$ for the function $g(x) = \ln x$.

8. (a) Consider the function $f(x) = \lim_{y\to\infty} x^y$, for $0 \le x \le 1$. At what point(s) is $f(x)$ discontinuous?

(b) Consider the function $f(x) = \lim_{y\to\infty} \frac{x^y}{x^y - 1}$, for $x \ge 0$. At what point(s) is $f(x)$ discontinuous?

9. Consider the function $f(x) = x^m$ where m is an integer, with the convention that $0^0 = 1$. What are the condition(s) on m that indicate whether $f(x)$ is continuous at $x = 0$?

10. A function $f(x)$ is said to have a removable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists, but either $f(a)$ is not defined or $\lim_{x\to a} f(x) \neq f(a)$.

(a) State the (exact) conditions needed for a rational function to have a removable discontinuity at $x = a$.

(b) Given that the rational function $\frac{f(x)}{g(x)}$ has a removable discontinuity at $x = a$, find $h(x)$ such that:

1) $h(x) = \frac{f(x)}{g(x)}$ $(x \neq a)$
2) $h(x)$ does not have a removable discontinuity at $x = a$.

11. Let $f(x) = \frac{|x^2-1|}{x^2-1}$.

(a) Explain why $f(x)$ is continuous wherever it is defined.

(b) For each point where $f(x)$ is not defined, state whether a value can be assigned to $f(a)$ in such a way as to make f continuous at a.