Supplementary Questions for HP Chapter 15

1. Derive the formula $\int \ln(x+10)\,dx = (x+10)\ln(x+10) - x + C$ in three ways:

(a) by substituting $u = x + 10$ and applying the result on page 869 on the text,

(b) integrating by parts with $u = \ln(x+10)$, $dv = dx$, $v = x$, and

(c) integrating by parts with $u = \ln(x+10)$, $dv = dx$ and $v = x + 10$.

2. Find the area of the region bounded by the curves $y = xe^{3x}$, $y = \frac{2}{3}xe^{x^2}$ and the lines $x = 0$ and $x = 3$.

3. **(a)** Show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

(b) Use the above formula to find $\int x^5 e^x\,dx$.

4. If we set $u = \frac{1}{x}$, $dv = dx$, $du = -\frac{1}{x^2}dx$, $v = x$, then we get

$$\int \frac{1}{x}\,dx = \left(\frac{1}{x}\right)bx - \int x\left(-\frac{1}{x^2}\,dx\right) = 1 + \int \frac{1}{x}\,dx$$

so

$$\int \frac{1}{x}\,dx = 1 + \int \frac{1}{x}\,dx$$

Subtracting $\int \frac{1}{x}\,dx$ from both sides gives

$$0 = 1.$$

What is wrong with this argument?

5. We have seen that the present value of A dollars t years from now is given by Ae^{-rt} where r is the annual rate of continuous compounding.

Suppose revenue of a company flows at a time-dependent rate $R(t)$. $R(t)$ tends to increase when business is good and tends to decrease when business is bad. The present value of an n-year revenue stream is given by $PV = \int_0^n R(t)e^{-rt}\,dt$, i.e., for n years the revenue is deposited, the amount given by $R(t)$ at any instant t. The above formula gives the present value of all revenue deposited into an account up to year n.

Let $R(t) = 1000 + 6t$.

(a) What is the present value of the first two years of revenue at 5%?

(b) at 10%?

(c) What is the present value of the first three years at 5%?

6. **(a)** Find $\int \frac{dx}{\sqrt{x}-\sqrt[3]{x}}$. Hint: try the substitution $u = \sqrt[6]{x}$.

(b) Find $\int \frac{\sqrt{x+4}}{x}\,dx$. Hint: try the substitution $u = \sqrt{x+4}$.

7. Show that for $n \geq 1$,

$$\int \frac{(x-1)^{n+1}-(x+1)^{n+1}}{(x^2-1)^{n+1}}\,dx = -\frac{1}{n}\left[\frac{(x-1)^n-(x+1)^n}{(x^2-1)^n}\right]+C$$

8. In the following two problems, make a preliminary substitution before using the method of partial fractions.

(a) $\int \frac{e^{4t}}{(e^{2t}-1)^3}\,dt$

(b) $\int \frac{1+\ln t}{t(3+2\ln t)^2}\,dt$

9. A store has an inventory of q units of a certain product at time $t = 0$. The store sells the product at a steady rate of $\frac{q}{w}$ units per week, exhausting the inventory in w weeks.

Find the average inventory level during the period $0 \leq t \leq w$. Does this agree with common sense?

10. The mean value theorem for definite integrals states if $f(x)$ is continuous on $[a, b]$, then there exists at least one number c between a and b such that

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx = f(c)$$

In other words, there exists at least one c with $a < c < b$ such that $f(c)$ is the average value of $f(x)$ over the interval $[a, b]$.

For the following two functions, find all values of c that satisfy the above theorem.

(a) $f(x) = x(x+1)$, $0 \leq x \leq 2$

(b) $f(x) = \frac{1}{x} - \frac{1}{x^2}$, $1 \leq x \leq e$

11. Two substances, A and B react to form a third substance C in such a way that if 30 grams of A and 20 grams of B are brought together at time $t = 0$, then the amount $x(t)$ of C present in the mixture has a rate of change with respect to time given by

$$\frac{dx}{dt} = k(30-x)(20-x), \qquad k > 0 \quad \text{is a constant and } x < 20$$

Solve for x as a function of t, assuming $x(0) = 0$.

12. For the following differential equations, find the equation of a solution which passes through the given point.

(a) $\frac{dy}{dx} = e^{x-y}$, $y(0) = 1$.

(b) $\frac{dy}{dx} = \frac{0.2y(18+0.1x)}{x(100+0.5y)}$, $y(10) = 10$ (Don't solve for y in this case.)

(c) $\frac{dy}{dx} = (1 + \ln x)y$, $y(1) = 1$.

13. Let $u(x)$ be a utility function for wealth. This means $u(x)$ is a measure of the satisfaction of owning x dollars in wealth. A utility function of constant relative risk aversion satisfies the differential equation

$$u''(x) = -\frac{u'(x)b}{x} \qquad (b \text{ is a constant})$$

Beginning with the substitution $v(x) = u'(x)$, solve for $u(x)$. (Assume $u'(x) > 0$ and $x > 0$.)

14. (a) Continuous compounding in a bank account at an interest rate of r per year can be modeled by the differential equation $\frac{dB}{dt} = rB$ where B is the balance in the account. Solve this differential equation if P is the principal in the account at time $t = 0$. (Write $\frac{1}{B}dB = r\,dt$.)

(b) If payments are made out of the account at a continuous rate of N dollars, then the differential equation becomes $dB/overdt = rB - N$. Solve this equation if P is the principal at time $t = 0$. (Write $\frac{1}{B-\frac{N}{r}}\,dB = r\,dt$.)

(c) Let $r = 0.05$, $N = 200$. Sketch the solution to (b) on the same set of axes for $P = 3000, 4000, 5000$.

15. In a particular country, beginning from time $t = 0$ (in years), interest rates increased according to the function $r = 0.25t + .50$, where r is the interest rate at time t. The rate of change of the balance B in a bank account was described by

$$\frac{dB}{dt} = (0.25t + 0.50)B$$

(a) Assuming $B = 100000$ when $t = 0$, find B as a function of t. (Assume $B > 0$.)

(b) How much money was in the account when t was five years?

16. A certain commodity is being sold at a price of \$$p$ per unit. Over a period of time, market forces will make this price tend towards the equilibrium price \$$p_0$, at which supply exactly balances demand. The rate at which the price changes is described by the Evans Price Adjustment model, which says that $\frac{dp}{dt}$ is proportional to the difference between the market price and the equilibrium price, that

$$\frac{dp}{dt} = k(p - p_0), \qquad k < 0 \quad \text{is a constant}$$

(a) Solve this equation for p as a function of t. (Assume $p \neq p_0$ for all t.)

(b) What happens to p as $t \to \infty$?

(c) What is the price when $t = 0$?

17. The gamma function is defined for all $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt.$$

(a) Find $\Gamma(1)$.

(b) Integrate by parts with respect to t to show that, for positive n, $\Gamma(n+1) = n\Gamma(n)$. (Assume $\lim_{a\to\infty} \frac{a^n}{e^a} = 0$ for all n.)

(c) Find a simple expression for $\Gamma(n)$ is n is a positive integer.

18. It is possible to determine whether an improper integral converges or diverges by comparing it to a known convergent or divergent integral.

If $\int_a^\infty f(x)\, dx$ is convergent and $|f(x)| \geq |g(x)|$ for all $x \geq a$, then $\int_a^\infty g(x)\, dx$ is convergent.

If $\int_{-\infty}^a f(x)\, dx$ is convergent and $|f(x)| \geq |g(x)|$ for all $x \leq a$, then $\int_{-\infty}^a g(x)\, dx$ is convergent.

If $\int_a^\infty f(x)\, dx$ is divergent and $|f(x)| \leq |g(x)|$ for all $x \geq a$, then $\int_a^\infty g(x)\, dx$ is divergent.

If $\int_{-\infty}^a f(x)\, dx$ is divergent and $|f(x)| \leq |g(x)|$ for all $x \leq a$, then $\int_{-\infty}^a g(x)\, dx$ is divergent.

Compare each of the following integrals to another integral in order to determine whether or not they converge. (Do not attempt to determine their value.)

(a) $\int_2^\infty \frac{x^2}{\sqrt{x^2-1}}\, dx$

(b) $\int_{-\infty}^{-2} \frac{\sqrt{-x}}{(x^2+5)^2}\, dx$

19. For what values of p is the integral $\int_1^\infty x^p\, dx$ convergent? If it is convergent, what is its value?

20. The rate, r, at which people get sick during an epidemic of the flu can be approximated by

$$r = 1000te^{-0.5t}$$

where r is measured in people per day and t is measured in days since the epidemic began.

(a) Sketch a graph of r as a function of $t \geq 0$. Assume $\lim_{t\to\infty} r(t) = 0$.

(b) When are people getting sick fastest?

(c) How many people get sick altogether?

21. On Purchase Timing for a Rapidly Improving Consumer's Good

In this question it is assumed that inflation is at an annual rate of r compounded continuously. In other words, the present value of M dollars t years in the future is $P = Me^{-rt}$.

Now, rapidly improving consumers' goods (the most obvious example being computers) have, for the purposes of this question, two main features:

(1) Because of technical progress, a version of a product becomes increasingly obsolete as time passes. Hence, the version's price decreases as time passes. It is assumed that the price of any version will be Ce^{-wt}, where C is a constant and w is referred to as the 'rate of cost reduction through technical progress'. For example, if $w = 1$ for version A, then if A used to cost \$5,000 at time $t = 0$ (i.e, $C = 5000$), then A will only cost $5000e^{-w(1)} = 5000e^{-1} = \frac{5000}{e}$ at time $t = 1$.

(2) Later versions of a product can generate more revenue per unit of time than earlier versions of the product. We assume in this question that any fixed version of a product generates a *constant* amount of revenue per unit of time.

(a) Suppose that a consumer with a side business wishes to upgrade a product. The earlier version generates a constant \$$R$ per unit of time and the newer version generates a constant \$$S$ per unit of time, where $S > R$. Also, the price of the 'newer version' is Ce^{-wt} at time t, for constants C and w. The consumer would like to choose the time T^* of purchase so as to optimize profits. BY finding, for any T, the present value (i.e, in terms of time $t = 0$) of profits if the consumer buys the new version at time T, find the time T^* that optimizes the present value of profits.

Note: assume that

1) the consumer will never upgrade again, and
2) the business will last 'indefinitely' (i.e., her business, for the purposes of simplicity, will last forever).

You may assume that any critical value is indeed a maximum. Hint: see ‘Integration as Applied to Annuities’ in §7.3 of HP (p. 883). Think of the constant revenue streams R and S as continuous annuities that are constant.

(b) A particular consumer with her own side photocopying business has decided to upgrade from her current SX-35A Copout to the new improved SX-35B Copycat. The Copout generates a constant \$600/month, whereas the Copycat generates a constant \$700/month. Inflation is at 12% annually, (but compounded continuously). If, at the beginning of 1996, the Copycat was $\$5,000$ but at the beginning of February, 1996 it was $\$4,901$, when should she stop Copping out and buy the Cat?