FACULTY OF ARTS AND SCIENCE
University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2005

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. *[3 marks]*

The market price of a \$100 bond with an annual coupon rate of 4%, paying semi-annually, which has 15 semi-annual coupons left, and which has an annual yield to maturity of 5%, is

Ⓐ \$66.86

Ⓑ \$76.81

Ⓒ \$89.87

Ⓓ \$93.81

Ⓔ \$97.85

2. *[3 marks]*

If $A = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $B^T AB =$

Ⓐ $[-1]$

Ⓑ $[0]$

Ⓒ $[2]$

Ⓓ $[-2]$

Ⓔ $[1]$

3. *[3 marks]*

The solution of the system

$$2x + 3y - 5z = 4,$$
$$4x + 5y - 7z = 10.$$

in terms of the parameter z, has $x =$

Ⓐ $-z + 4$

Ⓑ $3z - 2$

Ⓒ $-3z + 1$

Ⓓ $2z - 3$

Ⓔ $-2z + 5$

4. *[3 marks]*

The solution to the inequality $\dfrac{\ln x}{x - 2} \geq 0$ is given by

Ⓐ $0 \leq x \leq 1$ or $x \geq 2$

Ⓑ $x < 1$ or $x > 2$

Ⓒ $0 < x \leq 1$ or $x > 2$

Ⓓ $x \leq 0$ or $x > 2$

Ⓔ $x \leq 1$ or $x > 2$

5. *[3 marks]*

The function $f(x) = x^2 e^{-x}$

Ⓐ increases on $(0, \infty)$

Ⓑ has an absolute maximum at $x = 2$

Ⓒ increases on $(-\infty,\ 0)$

Ⓓ increases on $(0,\ 2)$

Ⓔ has no horizontal asymptote

6. *[3 marks]*

The line which is tangent to the graph of $y = (2x)^{x^2}$ at the point $(\frac{1}{2}, 1)$ has equation:

Ⓐ $8y - 2x = 7$

Ⓑ $4y - 2x = 3$

Ⓒ $y = 2x$

Ⓓ $2y - 2x = 1$

Ⓔ $4y + 2x = 5$

7. *[3 marks]*

The average value, on $[-4, 2]$, of $f(x) = |x^3|$ is

Ⓐ $\frac{34}{3}$

Ⓑ 10

Ⓒ -10

Ⓓ 68

Ⓔ -60

8. *[3 marks]*

For 10 years, cash flows into an account at the constant rate of \$1,000 per year. To the nearest dollar, the present value of the cash flow, if the account earns 4% compounded continuously, is

Ⓐ \$9,138

Ⓑ \$6,097

Ⓒ \$7,515

Ⓓ \$8,242

Ⓔ \$8,836

9. *[3 marks]*

The demand functions for the two products toves (T) and mome raths (M) are as follows:

$$q_T = \frac{1}{p_T + p_M} + c_1 p_M$$
$$q_M = \frac{1}{p_T + p_M} + c_2 p_T$$

where p_T is the price of one tove and p_M is the price of one mome rath and c_1 and c_2 are constants.

At the moment $p_T = \$2000$ and $p_M = \$1000$. For which of the following pairs of constants c_1 and c_2 are toves and mome raths competitive products (i.e., substitutes) at these prices?

Ⓐ $c_1 = 1,\ c_2 = 2$

Ⓑ $c_1 = -.00001,\ c_2 = -.00002$

Ⓒ $c_1 = .00000000001,\ c_2 = .00000000001$

Ⓓ $c_1 = -1,\ c_2 = -1.5$

Ⓔ $c_1 = -1,\ c_2 = 0.5$

10. *[3 marks]*

The partial derivative $\dfrac{\partial f}{\partial x}$ of $f(x, y, z) = e^{x^2 - yzx}$ is

Ⓐ $e^{x^2 - zx}$

Ⓑ $e^{2x - yz}$

Ⓒ $e^{x^2 - zx} \ln(zx - yz)$

Ⓓ $e^{x^2 - yzx}(2x - yz)$

Ⓔ $e^{2x - yz}(2x - yz)$

11. *[3 marks]*

If $f(x, y, z) = 2x^3 - 3y^2 z + 8xy^3 z^4 - 6xyz + 3xy^2$ then $f_{xyz}(1, 1, 1) =$

Ⓐ 90

Ⓑ 102

Ⓒ 24

Ⓓ 18

Ⓔ -4

12. *[3 marks]*

If $z = a^2 + 2b^2 - 4ac$ where $a = 2t,\ b = 4s - 3t,\ c = 2st^2$ then, when $s = t = 1$,

$\dfrac{\partial z}{\partial t} =$

Ⓐ -36

Ⓑ 12

Ⓒ -52

Ⓓ -28

Ⓔ 42

13. *[3 marks]*

If $z = (x^2 + y^2)^{10}$ where $x = 4r^2 s^3$ and $y = e^{2r+3s-3}$, then when $r = 0$ and $s = 1$,

$\dfrac{\partial z}{\partial r} =$

Ⓐ 40

Ⓑ 20

Ⓒ 0

Ⓓ 10

Ⓔ 1

14. *[3 marks]*

If $x^2 + xy + yz + z^2 = 6$ defines z implicitly as a function of x and y, then

$\dfrac{\partial z}{\partial y} =$

Ⓐ $-\dfrac{y + 3z}{y}$

Ⓑ $-\dfrac{x + 2z}{y}$

Ⓒ $-\dfrac{x + 2z}{x + 2y}$

Ⓓ $-\dfrac{x + z}{2z}$

Ⓔ $-\dfrac{x + z}{y + 2z}$

15. *[3 marks]*

$$\int_{-1}^{1}\int_{y}^{y^2}(2x+3y)\,dx\,dy =$$

Ⓐ $-\dfrac{11}{3}$

Ⓑ $-\dfrac{34}{15}$

Ⓒ $-\dfrac{9}{16}$

Ⓓ $-\dfrac{4}{3}$

Ⓔ $-\dfrac{2}{5}$

PART B. WRITTEN-ANSWER QUESTIONS

B1. *[10 marks]*

You wish to purchase a mine which will produce an annual return of \$32,000 per year for 12 years, after which the mine will have no value. At the end of each year, you plan to place money in a sinking fund earning 6.2% compounded annually, so as to replace the purchase price exactly after 12 years. If you want to earn 8% annually on this investment, what purchase price should you pay?

[Hint: Remember to include the replaced purchase price as part of the total return.]

B2.

(a) (i) *[2 marks]*

Find a so that the following limit exists

$$\lim_{x \to 0} \frac{e^{2x+1} - e - ax}{x^2}$$

(ii) *[5 marks]*

Find a and b so that the following limit exists, **then find the limit.**

$$\lim_{x \to 0} \frac{e^{2x+1} - e - ax - bx^2}{x^3}$$

B2.

(b) *[6 marks]*

Solve the initial value problem

$$y(2) = 1$$
$$(x-1)yy' = 1$$

then find $y(3)$.

B3. *[12 marks]*

Find the following integrals.

(a) *[6 marks]*

$$\int_0^{\infty} xe^{-3x}\, dx$$

(b) *[6 marks]*

$$\int \frac{x-14}{x^2-4}\, dx$$

B4. *[10 marks]*

Find and classify the critical points of the function given by:

$$f(x,y) = -x^3 + 3xy^2 + 12y^2 + 4y^3$$

B5. *[10 marks]*

A consumer has $600 to spend on two products which cost $20 per unit and $30 per unit respectively. The utility derived by the consumer from x units of the first product and y units of the second product is given by the utility function

$$U(x,y) = 10x^{0.6}y^{0.4}$$

[Note: the utility function measures the total satisfaction received by the consumer.]

Use the method of Lagrange multipliers (no marks for any other method) to find how many units of each product the consumer should buy to maximize his/her utility.

[You do **not** have to show that your answer does in fact give a maximum.]

Solutions to April 2005 Exam, MAT133Y

PART A

1. ANSWER: (D)

$$r = 0.02 \qquad i = 0.025 \qquad n = 15$$
$$p = 100(1.025)^{-15} + 2a_{\overline{15}|.025}$$
$$p = 93.809.$$

2. ANSWER: (C)

$$B^T AB = [1, 1] \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$= [1, 1] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [2]$$

3. ANSWER: (E)

$$\left[\begin{array}{ccc|c} 2 & 3 & -5 & 4 \\ 4 & 5 & -7 & 10 \end{array}\right]$$

$$\begin{array}{l} R_1 \to \frac{1}{2}R_1 \\ R_2 \to R_2 - 4R_1 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{5}{2} & 2 \\ 0 & -1 & 3 & 2 \end{array}\right]$$

$$\begin{array}{l} R_2 \to -R_2 \\ R_1 \to R_1 - \frac{3}{2}R_2 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & -2 \end{array}\right]$$

$$x + 2z = 5$$
$$x = -2z + 5$$

More simply

$$10x + 15y - 25z = 20$$
$$12x + 15y - 21z = 30$$
$$2x + 4z = 10 \Longrightarrow 2x = 10 - 4z$$
$$x = 5 - 2z$$

4. ANSWER: Ⓒ

ln x	-	+	+	
x-2	0 -	1 -	2 +	

$(0,1) \cup (2,\infty)$ give >0 and 1 gives 0. So $(0,\ 1] \cup (2,\infty)$

5. ANSWER: Ⓓ

$$\begin{aligned} f'(x) &= 2xe^{-x} - x^2e^{-x} \\ &= x(2-x)e^{-x} \end{aligned}$$

	f'	f
$(-\infty,0)$	-	dec
$(0,2)$	+	inc
$(2,\infty)$	-	dec

Note that the max at $x=2$ is only local $\left(\lim_{x\to-\infty} f(x) = \infty\right)$.
And $\lim_{x\to\infty} x^2e^{-x} = 0$ so there **is** a horizontal asymptote.

6. ANSWER: Ⓑ

$$\ln y = x^2 \ln 2x$$

$$\frac{1}{y}y' = 2x\ln 2x + \frac{x^2}{2x}\cdot 2$$

$$\text{at } (\tfrac{1}{2},\ 1),\ y' = 1.0 + \frac{1}{2} = \frac{1}{2}$$

$$y - 1 = \frac{1}{2}(x - \frac{1}{2}) = \frac{1}{2}x - \frac{1}{4}$$

$$y = \frac{1}{2}x + \frac{3}{4}$$

$$4y = 2x + 3$$

$$4y - 2x = 3$$

7. ANSWER: Ⓐ

$$\begin{aligned}\bar{f} &= \frac{1}{b-a}\int_a^b f(x)\,dx \\ &= \frac{1}{6}\int_{-4}^{2} |x^3|\,dx \\ &= \frac{1}{6}\Big[\int_{-4}^{0} -x^3\ dx + \int_0^2 x^3\,dx\Big] \\ &= \frac{1}{6}\Big[-\frac{x^4}{4}\Big|_{-4}^{0} + \frac{x^4}{4}\Big|_0^2\Big] \\ &= \frac{1}{6}[64+4] = \frac{68}{6} = \frac{34}{3}\end{aligned}$$

8. ANSWER: Ⓓ

$$\begin{aligned}\text{P.V.} &= \int_0^{10} 1000\,e^{-.04t}\,dt \\ &= -25,000\,e^{-.04t}\Big|_0^{10} \\ &= 25,000[1-e^{-.4}] \\ &\approx 8242\end{aligned}$$

9. ANSWER: Ⓐ

$$\begin{aligned}\frac{\partial q_T}{\partial p_M} &= -\frac{1}{(p_T+p_M)^2} + c_1 \\ \frac{\partial q_T}{\partial p_M} &= -\frac{1}{(3000)^2} + c_1 \\ \frac{\partial q_M}{\partial p_T} &= -\frac{1}{(3000)^2} + c_2\end{aligned}$$

We need both partials > 0, so c_1 and $c_2 > 0$. In fact

$$\begin{aligned}c_1 &> \frac{1}{(3000)^2} = \frac{1}{9\times 10^6} \\ c_2 &> \frac{1}{9\times 10^6}\end{aligned}$$

is what we need. Only Ⓐ does this.

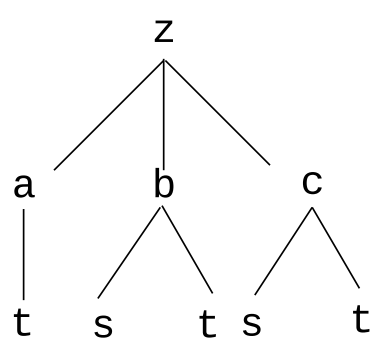
10. ANSWER: Ⓓ

$$\frac{\partial f}{\partial x} = e^{x^2 - yzx}(2x - yz)$$

11. ANSWER: Ⓐ

$$\begin{aligned} f_z &= -3y^2 + 32xy^3z^3 - 6xy \\ f_{zx} &= 32y^3z^3 - 6y \\ f_{zxy} &= 96y^2z^3 - 6 = 90 \text{ at } (1,1,1) \end{aligned}$$

12. ANSWER: Ⓒ



$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial a}\frac{da}{dt} + \frac{\partial z}{\partial b}\frac{\partial b}{\partial t} + \frac{\partial z}{\partial c}\frac{\partial c}{\partial t} \\ &= (2a - 4c)2 + 4b(-3) - 4c \cdot 4st \end{aligned}$$

At $s = t = 1$

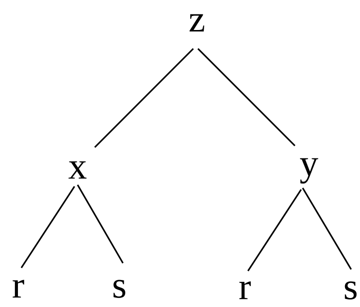
$a = 2$

$b = 1$ $\qquad = (4 - 8) \cdot 2 - 12 - 8.4$

$c = 2$

$= -52$

13. ANSWER: Ⓐ



$$r = 0, \qquad s = 1$$

$$x = 0, \qquad y = 1$$

$$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \\ &= 10(x^2+y^2)^9 \cdot 2 \times \cdot 8rs^3 + 10(x^2+y^2)^9 \cdot 2y \cdot e^{2r+35-3} \cdot 2 \\ &= 10 \cdot 2 \cdot 2 = 40\end{aligned}$$

14. ANSWER: Ⓔ

$$\begin{aligned} x + z + y\frac{\partial z}{\partial y} + 2z\frac{\partial z}{\partial y} &= 0 \\ \frac{\partial z}{\partial y} &= -\frac{x+z}{y+2z}\end{aligned}$$

15. ANSWER: Ⓑ

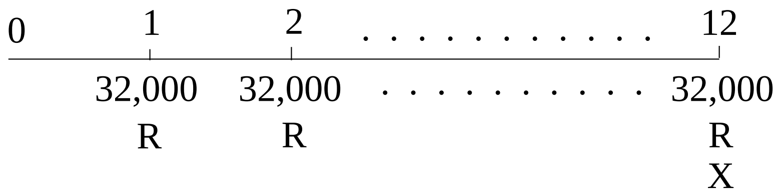
$$\int_{-1}^{1}\int_{y}^{y^2} (2x+3y)\,dx\,dy =$$

$$\begin{aligned} &= \int_{-1}^{1} \left[x^2 + 3xy\right]_{x=y}^{x=y^2} dy \\ &= \int_{-1}^{1} \left[(y^4+3y^3) - (y^2+3y^2)\right] dy \\ &= \left[\frac{y^5}{5} + y^4 - \frac{4y^3}{3}\right]_{-1}^{1} \\ &= \frac{2}{5} - \frac{8}{3} = -\frac{34}{15}\end{aligned}$$

PART B

B1.

Let X be the purchase price.
Let R be the annual payment to the sinking fund.



$$Rs_{\overline{12}|.062} = X$$

Analogous to a bond with face value X coupon $32,000 - R$ and yield to maturity $.08$ **and** price X.

$$X = X(1.08)^{-12} + (32,000 - R)a_{\overline{12}|.08}$$

Substituting $R = \dfrac{X}{s_{\overline{12}|.062}}$, this equation can be solved for X. More cleverly

$$\begin{aligned} X - X(1.08)^{-12} &= (32,000 - R)a_{\overline{12}|.08} \\ X\left(1 - (1.08)^{-12}\right) &= (32,000 - R)a_{\overline{12}|.08} \\ .08Xa_{\overline{12}|.08} &= (32,000 - R)a_{\overline{12}|.08} \end{aligned}$$

$.08X = 32,000 - R$: we could have started here. Since the purchase price is always being replaced, the 8% of purchase price need only cover the net annual income.

$$\begin{aligned} .08X + \frac{X}{s_{\overline{12}|.062}} &= 32,000 \\ X &= \frac{32,000}{.08 + \dfrac{1}{s_{\overline{12}|.062}}} \end{aligned}$$

$$\boxed{X \approx \$230,900}$$

B2.

(a) (i)

$$\lim_{x\to 0} \frac{e^{2x+1} - e - ax}{x^2} \qquad \frac{0}{0} \text{ no matter what } a \text{ is}$$

$$= \lim_{x\to 0} \frac{2e^{2x+1} - a}{2x}$$

can only have a limit if $\lim_{x\to 0} 2e^{2x+1} = a$ i.e. if

$$\boxed{a = 2e}$$

and then

$$= \lim_{x\to 0} \frac{4e^{2x+1}}{2} = 2e$$

so the limit does exist when $a = 2e$.

(a) (ii)

$$\lim_{x\to 0} \frac{e^{2x+1} - e - ax - bx^2}{x^3} \qquad \frac{0}{0} \text{ no matter what } a \text{ and } b \text{ are}$$

$$= \lim_{x\to 0} \frac{2e^{2x+1} - a - 2bx}{3x^2}$$ can only have a limit if

$$\lim_{x\to 0}(2e^{2x+1} - a - 2bx) = 0$$

i.e. $\boxed{a = 2e}$ as before

and then $= \lim_{x\to 0} \frac{4e^{2x+1} - 2b}{6x}$ can only have a limit if

$$\lim_{x\to 0}(4e^{2x+1} = 2b$$

i.e. $\boxed{b = 2e}$

and then $= \lim_{x\to 0} \frac{8e^{2x+1}}{6}$

$$= \boxed{\frac{4e}{3}}$$

B2.

(b)

$$
\begin{aligned}
(x-1)y\frac{dy}{dx} &= 1 \\
y\,dy &= \frac{dx}{x-1} \\
\frac{y^2}{2} &= \ln|x-1| + C \\
\frac{1}{2} &= \ln 1 + C \Rightarrow C = \frac{1}{2} \\
\frac{y^2}{2} &= \ln|x-1| + \frac{1}{2} \\
y^2 &= 2\ln|x-1| + 1 \\
y &= \boxed{\sqrt{\ln(x-1)^2+1}} \\
\text{(Note that)}\quad y &= -\sqrt{\ln(x-1)^2+1} \quad \text{gives} \quad y(2) = -1. \\
\text{or}\quad y &= \boxed{\sqrt{2\ln|x-1|+1}} \\
y(3) &= \boxed{\sqrt{2\ln 2+1}} \\
y(3) &\approx 1.5448
\end{aligned}
$$

B3.

(a) Let

$$
\begin{aligned}
&= \lim_{R\to\infty} \int_0^R xe^{-3x}\,dx \\
u &= x \qquad dv = e^{-3x}\,dx \\
du &= dx \qquad v = -\frac{e^{-3x}}{3} \\
&= \lim_{R\to\infty} \left[-\frac{xe^{-3x}}{3}\Big|_0^R + \frac{1}{3}\int_0^R e^{-3x}\,dx \right] \\
&= \lim_{R\to\infty} \left[-\frac{Re^{-3R}}{3} - \frac{1}{9}e^{-3x}\Big|_0^R \right] \\
&= \lim_{R\to\infty} \left[-\frac{Re^{-3R}}{3} - \frac{e^{-3R}}{9} + \frac{1}{9} \right]
\end{aligned}
$$

But by L'Hop or otherwise

$$\lim_{R\to\infty} Re^{-3R} = 0;\ e^{-3R} \to 0$$

as well.

$$\text{So} \qquad \lim_{R\to\infty} = \boxed{\frac{1}{9}}$$

B3.

(b)

$$\frac{x-14}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} = \frac{A(x+2)+B(x-2)}{x^2-4}$$
$$x - 14 = A(x+2) + B(x-2)$$

$$x = 2 \Rightarrow -12 = 4A \Rightarrow A = -3$$

$$x = -2 \Rightarrow -16 = -4B \Rightarrow B = 4$$

Alternatively solve

$$1 = A + B$$

$$-14 = 2A - 2B$$

to get

$$A = -3$$

$$B = 4$$

$$\int = \int \Big[-\frac{3}{x-2} + \frac{4}{4+2} \Big] dx = \boxed{-3\ln|x-2| + 4\ln|x+2| + C}$$

B4.

$$f_x = -3x^2 + 3y^2$$

$$f_y = 6xy + 24y + 12y^2$$

setting these to 0

$$y^2 = x^2 \Rightarrow y = x \quad \text{or} \quad y = -x$$

and

$$y(x + 4 + 2y) = 0$$

If $y = x,\ y(3y+4) = 0$, hence $y = 0$ or $y = -\frac{4}{3}$ Critical points are

$$\boxed{(0,0)} \qquad \text{and} \qquad \boxed{(-\frac{4}{3},\ -\frac{4}{3})}$$

If $y = -x,\ y(4+y) = 0\ y = 0$, or $y = -4$. Critical points are $(0,0)$ which we already have and $\boxed{(4,-4)}$

	$f_{xx} = -6x$	$f_{yy} = 6x + 24 + 24y$	$f_{xy} = 6y$
$(0,0)$	0	24	0
$(4,-4)$	-24	-48	-24
$\left(-\frac{4}{3},\ -\frac{4}{3}\right)$	8	-16	-8

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2$$

$$\boxed{D(0,0)} = 0 \ \boxed{\text{no info.}}$$

$$\boxed{D(4,-4)} = 24 \cdot 48 - 24^2 = 24^2 = 576 > 0 \text{ extremum and } f_{xx} = -24 < 0 \text{ so } \boxed{\text{local max.}}$$

$$\boxed{D(-\frac{4}{3}, -\frac{4}{3})} = 8(-16) - 8^2 = -64 < 0 \text{ so } \boxed{\text{no local extremum.}}$$

B5.

$$20x + 30y = 600$$
$$2x + 3y = 60$$
$$L = 10x^{0.6}y^{0.4} - \lambda(2x + 3y - 60)$$
$$L_x = 6x^{-0.4}y^{0.4} - 2\lambda = 0 \Rightarrow \lambda = 3\left(\frac{y}{x}\right)^{0.4}$$
$$L_y = 4x^{0.6}y^{-0.6} - 3\lambda = 0 \Rightarrow \lambda = \frac{4}{3}\left(\frac{x}{y}\right)^{0.6}$$
$$L_y = -(2x + 3y - 60) = 0 \Rightarrow 2x + 3y = 60$$
$$3\left(\frac{y}{x}\right)^{0.4} = \frac{4}{3}\left(\frac{x}{y}\right)^{0.6} \Rightarrow 3y = \frac{4}{3}x$$

So

$$2x + \frac{4}{3}x = 60 \Rightarrow \frac{10x}{3} = 60 \Rightarrow x = 18$$
$$3y = \frac{4}{3} \cdot 18 \Rightarrow y = 8$$

So

$$\boxed{x = 18,\ y = 8}$$