

FACULTY OF ARTS AND SCIENCE

University of Toronto

FINAL EXAMINATIONS, APRIL/MAY 2006

MAT 133Y1Y

Calculus and Linear Algebra for Commerce

PART A. MULTIPLE CHOICE

1. [3 marks]

The derivative of $x^3\sqrt{x^2+1}$ is:

- Ⓐ $\frac{3x^2}{2x\sqrt{x^2+1}}$
- Ⓑ $3x^2\sqrt{2x}$
- Ⓒ $3x^2\sqrt{x^2+1} + x^3\frac{1}{2\sqrt{x^2+1}}$
- Ⓓ $3x^2\sqrt{x^2+1} + x^3\sqrt{2x}$
- Ⓔ $3x^2\sqrt{x^2+1} + \frac{x^4}{\sqrt{x^2+1}}$

2. [3 marks]

$\lim_{x \rightarrow 1} \frac{2\sqrt{x} - x - 1}{(x-1)^2} =$

- Ⓐ $-\frac{1}{4}$
- Ⓑ 1
- Ⓒ $\frac{1}{2}$
- Ⓓ $-\frac{1}{2}$
- Ⓔ $\frac{1}{4}$

3. [3 marks]

The function $f(x) = \frac{x}{(x-1)^2}$ on the interval $(-\infty, 2]$ has

- Ⓐ an absolute minimum value of $-\frac{1}{4}$ and no absolute maximum value
- Ⓑ an absolute minimum value of -1 and no absolute maximum value
- Ⓒ an absolute minimum value of $-\frac{1}{4}$ and an absolute maximum value of 2
- Ⓓ an absolute minimum value of -1 and an absolute maximum value of 2
- Ⓔ no absolute maximum or minimum values

4. [3 marks]

$$\int_1^e \frac{3^{\ln(x^2)}}{x} dx =$$

- Ⓐ $8/\ln 3$
- Ⓑ $(9 - e)/e$
- Ⓒ 4
- Ⓓ $4/\ln 3$
- Ⓔ $4 \ln 3$

5. [3 marks]

If the demand equation for a product is $p = 300 - q^2$ and the supply equation is $p = 10q + 100$ then the Producers' Surplus at market equilibrium is

- Ⓐ 500
- Ⓑ 100
- Ⓒ 2500
- Ⓓ 0
- Ⓔ $2000/3$

6. [3 marks]

Given that: $f(1) = 1$, $f(2) = -2$, $f(3) = -1$, $f(4) = 3$, $f(5) = 4$;

if we use Simpson's Rule with $n = 4$, then the value of $\int_1^5 f(x) dx$ is approximately:

- Ⓐ 7
- Ⓑ $5/2$
- Ⓒ $7/3$
- Ⓓ $5/3$
- Ⓔ $7/2$

7. [3 marks]

$\int_1^3 x^2 \ln x dx$ is equal to

- Ⓐ $9 \ln 3 - \frac{26}{9}$
- Ⓑ $9 \ln 3$
- Ⓒ $9 \ln 3 - 4$
- Ⓓ $9 \ln 3 + \frac{26}{9}$
- Ⓔ $9 \ln 3 + 4$

8. [3 marks]

$\int \frac{x^2 + x + 1}{x^3 + x^2} dx$ is equal to:

- Ⓐ $\ln|x^3 + x^2| + C$
- Ⓑ $\ln|x + 1| - \frac{1}{x} + C$
- Ⓒ $\ln|x(x + 1)| + C$
- Ⓓ $\ln|x| + \ln|x^3 + x^2| + C$
- Ⓔ $\frac{2x + 1}{3x^2 + 2x} + C$

9. [3 marks]

$$\int_2^{\infty} \frac{x-1}{x^3} dx \text{ is}$$

- Ⓐ $\frac{1}{4}$
- Ⓑ $\frac{1}{2}$
- Ⓒ $\frac{7}{16}$
- Ⓓ $\frac{3}{8}$
- Ⓔ undefined; the integral diverges

10. [3 marks]

If products A and B have joint demand functions

$$q_A(p_A, p_B) = 128 - p_A + 8p_B - p_A p_B$$

$$q_B(p_A, p_B) = 288 - 6p_A - 18p_B + 2p_A p_B$$

then they are complementary products provided

- Ⓐ $p_A < 8$ and $p_B > 3$
- Ⓑ $p_A > 8$ and $p_B < 3$
- Ⓒ $p_A < 9$
- Ⓓ $p_A < 8$ and $p_B < 3$
- Ⓔ $p_A > 8$ and $p_B > 3$

11. [3 marks]

If $f(x, y) = xy$ where $x = x(t)$ and $y = y(t)$ satisfy

$$x(1) = 3 \quad \frac{dx}{dt}(1) = 4$$

$$y(1) = -2 \quad \frac{dy}{dt}(1) = 5,$$

then $\frac{df}{dt}(1) =$

- Ⓐ 2
- Ⓑ -8
- Ⓒ -10
- Ⓓ 7
- Ⓔ 12

12. [3 marks]

If $xyz = 1$, then $\frac{\partial z}{\partial x} =$

- Ⓐ $-\frac{z}{y}$
- Ⓑ $-\frac{y}{x}$
- Ⓒ $-\frac{x}{z}$
- Ⓓ $-\frac{x}{y}$
- Ⓔ $-\frac{z}{x}$

13. [3 marks]

If $g(u, v, w) = \frac{(u^2 + 3v)^2}{4w - 5}$, then at $u = -1$, $v = 1$, $w = 2$, $\frac{\partial^2 g}{\partial u \partial w} =$

- Ⓐ $\frac{4}{9}$
- Ⓑ $\frac{16}{3}$
- Ⓒ $\frac{8}{9}$
- Ⓓ $\frac{8}{3}$
- Ⓔ $\frac{64}{9}$

14. [3 marks]

The function $f(x, y) = \frac{1}{3}x^3 + \frac{1}{2}y^2 + xy - 6x + 3$ has a relative minimum at the point

- Ⓐ (2, 2)
- Ⓑ (-2, 2)
- Ⓒ (2, -2)
- Ⓓ (-3, 3)
- Ⓔ (3, -3)

15. [3 marks]

$$\int_0^1 \int_{-y}^y 21x^2y^3 dx dy =$$

- Ⓐ 0
- Ⓑ $\frac{7}{16}$
- Ⓒ 14
- Ⓓ $\frac{7}{14}$
- Ⓔ 2

PART B. WRITTEN-ANSWER QUESTIONS

B1. [12 marks]

Mr. Jones has just won enough at the lottery to retire 3 years early. To do so, he will need \$2,000 at the end of each month from January, 2010 to December, 2012 inclusive. At the beginning of 2005 he bought a GIC paying 8% effective annual interest and maturing at the beginning of 2010.

(a) [6 marks]

Assuming the following:

- (i) Mr. Jones will continue earning 8% effective annual for the foreseeable future after 2009 and
- (ii) his initial GIC investment was large enough to just cover the 36 \$2,000 withdrawals, how much did Mr. Jones invest at the beginning of 2005?

(b) [6 marks]

Suppose that, at the beginning of 2010, Mr. Jones finds that the best investment he can make which allows monthly withdrawals only yields 0.2% effective monthly interest. In this event, he plans to make a \$2,000 withdrawal at the end of January, 2010, and continue making a \$2,000 withdrawal at the end of each month until the money runs out. If he follows this plan (assuming his savings always earn 0.2% effective monthly interest after 2009), how many full \$2,000 withdrawals will he be able to make?

B2. [10 marks]

Consider the curves $y = e^x + x + a$ and $y = b + cx - x^2$, where a, b, c are real constants. Find a, b and c such that the two curves are tangent to each other at the point $(0, 2)$. (Two curves are tangent at a point if they intersect at that point **and** they have the same tangent line at that point.)

B3. [11 marks]

Find the area of the region bounded by the curves $y = xe^x$ and $y = \frac{x}{e^2}$.

[A sketch of the curves and the resulting region would probably be helpful.]

B4. [11 marks]

It is known that the function $f(x, y) = x - y$ subject to the constraint $x^2 + xy + y^2 = 100$ has both an absolute maximum and absolute minimum value.

Find the points at which f attains these extreme values. What are the absolute maximum and minimum values of f ?

B5. [11 marks]

The growth rate of a tumor satisfies $\frac{dV}{dt} = 0.2Ve^{-0.01t}$, where t is the time in months and V is the volume in cubic millimetres.

(a) [8 marks]

If the initial volume at $t = 0$ is V_0 , find V as a function of t .

(b) [3 marks]

Find the volume of the tumor after 6 months if the initial volume is 1.86 cubic millimetres.

Solutions to April 2006 Exam, MAT133Y
PART A

1. ANSWER: Ⓔ

$$\begin{aligned} & 3x^2\sqrt{x^2+1} + x^3 \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x \\ = & 3x^2\sqrt{x^2+1} + \frac{x^4}{\sqrt{x^2+1}} \end{aligned}$$

2. ANSWER: Ⓐ

$$\begin{aligned} & \text{L'Hopital } \frac{0}{0} \\ & \lim_{x \rightarrow 1} \frac{\frac{1}{\sqrt{x}} - 1}{2(x-1)} \quad \text{still } \frac{0}{0} \\ = & \lim_{x \rightarrow 1} \frac{-\frac{1}{2}x^{-\frac{3}{2}}}{2} = -\frac{1}{4} \end{aligned}$$

or by substitution $u = \sqrt{x}$ so $x \rightarrow 1 \Rightarrow u \rightarrow 1$

$$\begin{aligned} & \lim_{u \rightarrow 1} \frac{2u - u^2 - 1}{(u^2 - 1)^2} \\ = & \lim_{u \rightarrow 1} \frac{-(u-1)^2}{(u-1)^2(u+1)^2} \\ = & \lim_{u \rightarrow 1} -\frac{1}{(u+1)^2} = -\frac{1}{4} \end{aligned}$$

3. ANSWER: Ⓐ There is no absolute max since $\lim_{x \rightarrow 1} \frac{x}{(x-1)^2} = +\infty$.

$$f'(x) = \frac{(x-1)^2 - x \cdot 2(x-1)}{(x-1)^4} = \frac{-(x+1)}{(x-1)^3}$$

	f'	f
$(-\infty, -1)$	-	dec
$(-1, 1)$	+	inc
$(1, \infty)$	-	dec

f decreases from $-\infty$ to -1 ; increases from -1 to 1 . Hence the min value of f on $(-\infty, 1)$ is at $x = -1$ where $f(x) = -\frac{1}{4}$.

The interval $(1, 2]$ is irrelevant since all value of $f(x)$ are positive (hence $> -\frac{1}{4}$) there.

4. ANSWER: Ⓓ

$$\text{Either let } u = \ln x^2 = 2 \ln x$$

$$du = \frac{2}{x} dx$$

$$\int = \int_0^2 3^u \frac{du}{2} = \frac{1}{2} \frac{3^u}{\ln 3} \Big|_0^2 = \frac{9-1}{2 \ln 3} = \frac{4}{\ln 3}$$

or

$$\frac{3^{\ln x^2}}{x} = \frac{3^{2 \ln x}}{x} = \frac{9^{\ln x}}{x} = \frac{e^{\ln x \ln 9}}{x} = \frac{(e^{\ln x})^{\ln 9}}{x} = \frac{x^{\ln 9}}{x}$$

so

$$\int = \int_1^e x^{\ln 9 - 1} dx = \frac{x^{\ln 9}}{\ln 9} \Big|_1^e = \frac{e^{\ln 9} - 1}{2 \ln 3} = \frac{9-1}{2 \ln 3} = \frac{4}{\ln 3}$$

5. ANSWER: Ⓐ

Equilibrium

$$300 - q^2 = 10q + 100$$

$$0 = q^2 + 10q - 200$$

$$0 = (q + 20)(q - 10)$$

$$q_0 = 10 \text{ (not } q = -20); p_0 = 200$$

$$\begin{aligned} \text{P.S} &= \int_0^{10} [200 - (10q + 100)] dq \\ &= \int_0^{10} (100 - 10q) dq \\ &= [100q - 5q^2]_0^{10} = 1000 - 500 = 500 \end{aligned}$$

6. ANSWER: Ⓒ

$$\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$$

x_0	x_1	x_2	x_3	x_4
1	2	3	4	5

$$\begin{aligned} S_4 &= \frac{\Delta x}{3} [y_0 + 4y_1 + 2y_2 + 4y_3 + y_4] \\ &= \frac{1}{3} [1 + 4(-2) + 2(-1) + 4(3) + 4] \\ &= \frac{7}{3} \end{aligned}$$

7. ANSWER: Ⓐ By parts

$$\begin{aligned}u &= \ln x & dv &= x^2 \\du &= \frac{dx}{x} & v &= \frac{x^3}{3} \\ \int &= \frac{x^3}{3} \ln x \Big|_1^3 - \frac{1}{3} \int_1^3 x^2 dx \\ &= \frac{3^3 \ln 3}{3} - 0 - \frac{1}{9} x^3 \Big|_1^3 \\ &= 9 \ln 3 - \frac{1}{9} (27 - 1) \\ &= 9 \ln 3 - \frac{26}{9}\end{aligned}$$

8. ANSWER: Ⓑ

$$\begin{aligned}\int \frac{x^2 + x + 1}{x^2(x+1)} dx \\ \frac{x^2 + x + 1}{x^2(x+1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \\ Ax(x+1) + B(x+1) + Cx^2 &= x^2 + x + 1 \\ x = 0 &\Rightarrow B = 1 \\ x = -1 &\Rightarrow C = 1 \\ x = 1 &\Rightarrow 2A + 2B + C = 3 \\ 2A + 3 &= 3 \Rightarrow A = 0 \\ \int &= \int \left[\frac{1}{x^2} + \frac{1}{x+1} \right] dx = -\frac{1}{x} + \ln|x+1| + C\end{aligned}$$

9. ANSWER: Ⓓ

$$\begin{aligned}&= \lim_{R \rightarrow \infty} \int_2^R \left(\frac{1}{x^2} - \frac{1}{x^3} \right) dx \\ &= \lim_{R \rightarrow \infty} \left(-\frac{1}{x} + \frac{1}{2x^2} \right)_2^R \\ &= \lim_{R \rightarrow \infty} \left(-\frac{1}{R} + \frac{1}{2R^2} \right) - \left(-\frac{1}{2} + \frac{1}{8} \right) \\ &= \frac{3}{8}\end{aligned}$$

10. ANSWER: Ⓑ

$$\frac{\partial q_A}{\partial p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} < 0$$

$$\frac{\partial q_A}{\partial p_B} = 8 - p_A < 0 \Rightarrow p_A > 8$$

$$\frac{\partial q_B}{\partial p_A} = -6 + 2p_B < 0 \Rightarrow p_B < 3$$

11. ANSWER: Ⓓ

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= y \frac{dx}{dt} + x \frac{dy}{dt} \\ &= -2 \cdot 4 + 3 \cdot 5 \quad \text{at } t = 1 \\ &= 7 \end{aligned}$$

12. ANSWER: Ⓔ

Either implicitly ($z = f(x, y)$)

$$\begin{aligned} yz + xy \frac{\partial z}{\partial x} &= 0 \\ xy \frac{\partial z}{\partial x} &= -yz \\ \frac{\partial z}{\partial x} &= -\frac{yz}{xy} = -\frac{z}{x} \end{aligned}$$

or

$$\begin{aligned} z &= \frac{1}{xy} \\ \frac{\partial z}{\partial x} &= -\frac{1}{x^2 y} = -\frac{1}{x} \cdot \frac{1}{xy} \\ &= -\frac{1}{x} z \\ &= -\frac{z}{x} \end{aligned}$$

13. ANSWER: ⑤

$$\begin{aligned}\frac{\partial g}{\partial w} &= \frac{-4(u^2 + 3v)^2}{(4w - 5)^2} \\ \frac{\partial^2 g}{\partial u \partial w} &= \frac{-4 \cdot 2(u^2 + 3v) \cdot 2u}{(4w - 5)^2} \\ &= \frac{-16u(u^2 + 3v)}{(4w - 5)^2} \\ &= \frac{-16(-1)(4)}{9} \quad \text{at } (-1, 1, 2) \\ &= \frac{64}{9}\end{aligned}$$

14. ANSWER: ⑤

$$\begin{aligned}f_x &= x^2 + y - 6 = 0 \\ f_y &= y + x = 0 \Rightarrow y = -x \\ x^2 - x - 6 &= 0 \\ (x - 3)(x + 2) &= 0\end{aligned}$$

$$\boxed{x = 3, \quad y = -3}$$

$$\boxed{x = -2, \quad y = 2}$$

$$f_{xx} = 2x \quad f_{yy} = 1 \quad f_{xy} = 1$$

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 2x - 1$$

$$D(3, -3) = 5 > 0 \quad \text{so local extremum}$$

$$f_{xx} = 6 > 0 \quad \text{so local min.}$$

$$D(-2, 2) = -5 < 0 \quad \text{no local extremum.}$$

15. ANSWER: ⑤

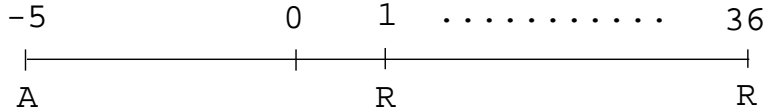
$$\begin{aligned}& \int_0^1 7x^3 y^3 \Big|_{x=-y}^{x=y} dy \\ &= \int_0^1 7y^3 [y^3 - (-y)^3] dy \\ &= \int_0^1 7y^3 \cdot 2y^3 dy \\ &= 14 \int_0^1 y^6 dy \\ &= 2\end{aligned}$$

PART B

B1.

(a)

Let r = monthly rate. Then $(1+r)^{12} = 1.08$. There are 36 required payments of \$2000 each. Let A be the initial investment



$$A(1.08)^{-5} = 2000 a_{\overline{36}|r}$$

$$A = (1.08)^{-5} \cdot 2000 \frac{[1 - (1+r)^{-36}]}{r} = \frac{(1.08)^{-5} \cdot 2000 [1 - (1.08)^{-3}]}{(1.08)^{\frac{1}{12}} - 1}$$

$$= \boxed{\$43,616.31}$$

(b)

His money has accumulated just the same, to $A(1.08)^5 = \$64,086.67$ but now $r = .002$:

$$64,086.67 = 2000 a_{\overline{n}|.002} = 2000 \frac{[1 - (1.002)^{-n}]}{.002}$$

$$1 - (1.002)^{-n} = \frac{64,086.67 \times .002}{2000} = 64,086.67 \times 10^{-6}$$

$$(1.002)^{-n} = 1 - 64,086.67 \times 10^{-6}$$

$$n = \frac{-\ln[1 - 64,086.67 \times 10^{-6}]}{\ln(1.002)} \approx 33.15$$

so $\boxed{33 \text{ full payments}}$

B2.

To intersect at $(0, 2)$:

$$2 = e^0 + 0 + a \quad \text{so} \quad \boxed{a = 1}$$

$$\text{and } 2 = b + c \cdot 0 - 0 \quad \text{so} \quad \boxed{b = 2}$$

$$y' = e^x + 1 \quad \text{for the first curve}$$

$$y' = c - 2x \quad \text{for the second curve } cr$$

At $(0, 2)$ these must give the same value:

$$e^0 + 1 = c \quad \text{so} \quad \boxed{c = 2}$$

B3.

Intersection at

$$xe^x = xe^{-2} \quad x(e^x - e^{-2}) = 0$$

The curves intersect at $x = 0$ and $x = -2$.

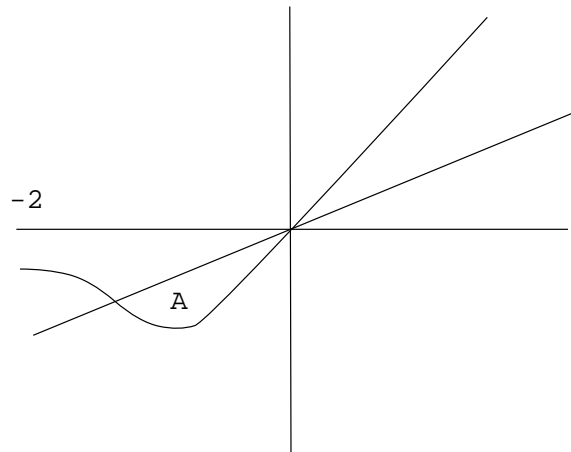
The bounded region must be from $x = -2$ until $x = 0$.

At

$$\begin{array}{llll} x = -1 & y = xe^x & \text{has} & y = -\frac{1}{e} \\ & y = \frac{x}{e^2} & \text{has} & y = -\frac{1}{e^2} \end{array}$$

Since $\frac{1}{e^2} < \frac{1}{e}$, $-\frac{1}{e} < -\frac{1}{e^2}$

So $y = \frac{x}{e^2}$ lies above $y = xe^x$ on $[-2, 0]$.



$$\begin{aligned} A &= \int_{-2}^0 \left[\frac{x}{e^2} - xe^x \right] dx \\ &= \frac{x^2}{2e^2} \Big|_{-2}^0 - \int_{-2}^0 xe^x dx \quad \begin{array}{l} u = x \\ du = dx \end{array} \quad \begin{array}{l} dv = e^x dx \\ v = e^x \end{array} \\ &= -\frac{2}{e^2} - \left[xe^x \Big|_{-2}^0 - \int_{-2}^0 e^x dx \right] \\ &= -\frac{2}{e^2} - \left[-\left(\frac{-2}{e^2}\right) - e^x \Big|_{-2}^0 \right] \\ &= -\frac{4}{e^2} + \left(e^0 - \frac{1}{e^2} \right) \\ &= \boxed{1 - \frac{5}{e^2}} \\ &\approx .3233 \end{aligned}$$

B4.

$$\mathcal{L} = x - y - \lambda(x^2 + xy + y^2 - 100)$$

$$\mathcal{L}_x = 1 - 2\lambda x - \lambda y = 0$$

$$\mathcal{L}_y = -1 - \lambda x - 2\lambda y = 0$$

$$\mathcal{L}_\lambda = 0 : x^2 + xy + y^2 = 100$$

$$2\lambda x + \lambda y = 1$$

$$\lambda x + 2\lambda y = -1$$

$$4\lambda x + 2\lambda y = 2$$

$$\frac{\lambda x + 2\lambda y = -1}{3\lambda x = 3 \quad \text{so} \quad \lambda \neq 0}$$

$$x = \frac{1}{\lambda}$$

$$2\lambda x + \lambda y = 1$$

$$2\lambda x + 4\lambda y = -2$$

$$3\lambda y = -3$$

$$y = -\frac{1}{\lambda}$$

Using

$$L_\lambda = 0$$

$$\frac{1}{\lambda^2} - \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 100$$

$$\frac{1}{\lambda^2} = 100$$

$$\lambda = \pm \frac{1}{10}$$

$$\lambda = \frac{1}{10} \quad x = 10 \quad y = -10 \quad f(x, y) = 20$$

$$\lambda = -\frac{1}{10} \quad x = -10 \quad y = 10 \quad f(x, y) = -20$$

Hence (10, -10) gives the max value of 20

and (-10, 10) gives the min value of -20

B5.

(a)

$$\begin{aligned}\frac{dV}{V} &= .2e^{-.01t} \\ \ln V &= -20e^{-.01t} + C \\ V &= Ae^{-20e^{-.01t}} \\ V_0 &= Ae^{-20} \quad \text{so} \quad A = V_0e^{20} \\ V &= V_0e^{20-20e^{-.01t}}\end{aligned}$$

$$\boxed{V = V_0e^{20(1-e^{-0.01})}}$$

(b)

$$\begin{aligned}V(6) &= 1.86e^{20(1-e^{-.06})} \\ &\approx \boxed{5.96} \text{ cubic millimetres}\end{aligned}$$