2. Bond Prices

A bond is a security which offers semi-annual* interest payments, at a rate r, for a fixed period of time, followed by a return of capital.

Suppose you purchase a $1,000 utility bond, freshly issued, which offers 16% interest per annum, payable semi-annually, and matures in 15 years. What you have in effect purchased is an annuity of $15 \times 2 = 30$ semi-annual interest payments of $80 each (8% of $1,000), plus a return of your $1,000 15 years later. On the face of it, you are receiving $3,400 in return for your original $1,000. But the interest payments span a period of 15 years, so let us calculate the present value of all interest payments, and the present value of the $1,000 returned in 15 years' time.

The present value will depend on the current interest rate; suppose it is i per semi-annual period. Then the present value of the above bond is given by the present value of all 30 interest payments, which is $80a_{\overline{30}|i}$, plus the present value of the $1,000 received in 15 years, which is $1,000(1+i)^{-30}$. Hence the present value of the bond is

$$P = 1000(1+i)^{-30} + 80a_{\overline{30}|i} \, .$$

Note that if $i = .08$,

$$P = 99.38 + 900.62$$
$$= 1000 \; ;$$

but if $i < .08$, P is greater than $1,000$; and if $i > .08$, P is less that $1,000$.

In general we can show that for a bond with face value V, with n outstanding interest payments at rate r each,

(1) $$P = V(1+i)^{-n} + rVa_{\overline{n}|i} \, ,$$

where i is the current interest rate per semi-annual period. In formula (1), P is referred to as the price of the bond, r the coupon rate, and i the yield rate. (Coupon and yield rates are often quoted as nominal annual rates, namely $2r$ and $2i$.)

Since bonds are long-term securities, the interest rates at which they are issued, r, may eventually be quite different from future interest rates, i. Hence when bonds trade on the open market, they trade at a price reflecting the yield rate which the purchaser requires. The price is usually calculated according to formula (1). (Unless the purchase is made in between interest

---

* We shall assume semi-annual payments; there are other possibilities.

payments, so that accrued interest must be taken into account − but we shall not consider such cases. See, however, Exercise 5).

Example 1. In newspapers, bond prices are quoted per $100 face value. So for instance, a bond quoted in the *Globe and Mail* of August 1, 1982 as

$$\text{Canada} \quad 10\frac{1}{4} \quad \text{February 1-04} \quad 66.75 \quad 15.67$$

is a Government of Canada bond, offering an annual coupon rate of $10\frac{1}{4}\%$, maturing on February 1, 2004, trading at $66.75, offering a yield (annual) of 15.67%.

Let us see how closely the given figures satisfy formula (1): take $r = .05125$, $i = .07835$, $V = 100$, and $n = 43$ (why?) and substitute to find

$$\begin{aligned} P &= 100(1.07835)^{-43} + 5.125 \, a_{\overline{43}|.07835} \\ &= 66.76 \,, \end{aligned}$$

almost the exact price as quoted. Note that since $r < i$, this bond is selling for less than face value—it is said to be selling at a discount.

Or consider a corporate bond, quoted in the same issue of the *Globe and Mail* as,

$$\text{Nova} \quad 17\frac{3}{4} \quad \text{February 15-97} \quad 102.00 \quad 17.37 \,.$$

This bond offers an annual coupon rate of $17\frac{3}{4}\%$, matures on February 15, 1997, is selling at $102 and offers an annual yield of 17.37%. Taking $r = .08875$, $i = .08685$, $V = 100$ and $n = 29$, we find

$$\begin{aligned} P &= 100(1.08685)^{-29} + 8.875 \, a_{\overline{29}|.08685} \\ &= 101.99 \,, \end{aligned}$$

which is only 1¢ less than the price quoted.

Note that since $r > i$, this bond is selling for more than face value—it is said to be selling at a premium.

Example 2. The manager of an investment fund has $1,000,000.00 to invest in government guaranteed bonds. He can purchase this amount of a new issue of Government of Canada Bonds at face value; they are $10\frac{1}{4}\%$ bonds maturing in 20 years. He is also interested in the bonds of a provincial utility that are on the market now; they mature in 19 years and bear interest at the rate of 10%.

2

(a) What maximum price per \$100.00 should he offer for the provincial utility bonds?

(b) What total face value in integral multiples of \$100.00 can he purchase for a price not to exceed \$1,000,000.00?

Solution.

(a) The investor is presumably assuming that over a period of roughly 20 years, the $10\frac{1}{4}\%$ yield of the Canada bonds is the minimum he should get. So he wants at least $10\frac{1}{4}\%$ yields on the utility bonds. Hence his price for these bonds per \$100.00 is not over

$$
\begin{aligned}
P &= 100(1.05125)^{-38} &+& \quad 5\big(1-(1.05125)^{-38}\big)/.05125 \\
&= 14.96827 &+& \quad\quad\quad 82.95778 \\
&= \quad 97.92605 & &
\end{aligned}
$$

(b) Since $97.92605 \times 10211 = 999922.90$, he will purchase \$1,021,100.00 face value, and have \$77.10 left over.

Example 3. The problem of determining the yield, given the price of a bond, is both more useful and more difficult. It is necessary to solve

$$ P = V(1+i)^{-n} + rV\big(1-(1+i)^{-n}\big)/i $$

for i, given P (and V, r, n), which is not easy.

However, with the availability of hand calculators that can do exponentiation, it is rather easy to estimate i by "trial and error".

Consider a bond price that is quoted as 115.03 (per 100 of face value), has a coupon rate of 14% and has 10 years to maturity. What is the yield at this price?

We observe that $P > V(=100)$ so $i < r(=.07)$. Let us try $i = .06$ and calculate the price

$$ 100(1.06)^{-20} + 7\big(1-(1.06)^{-20}\big)/.06 = 111.47 . $$

This is less than 115.03, so let us try $i = .05$:

$$ 100(1.05)^{-20} + 7\big(1-(1.05)^{-20}\big)/.05 = 124.92 . $$

This is more than 115.03, so let us try $i = .055$. This gives a price of 117.93. If we try $i = .057$ we get 115.28. This may be close enough. However, $i = .0572$ gives a price of 115.02, so we conclude the yield is very close to 11.44% per annum.

Exercises

1. Calculate the price of the bond described on p. 6 if

$$i = .09; \qquad i = .07 .$$

2. (a) Find the price of the bonds or debentures given $V = 100$ and r, i, and n as stated.

   (i) 20 years to maturity, interest at 10% p.a. to yield 11% p.a.

   (ii) 2 years to maturity, interest at 10% p.a. to yield 11% p.a.

   (iii) 20 years to maturity, interest at 11% p.a. to yield 10% p.a.

   (iv) 2 years to maturity, interest at 11% p.a. to yield 10% p.a.

   (v) 12 years to maturity, interest at $12 \frac{1}{2}$ % p.a. payable annually to yield $14 \frac{1}{4}$ % p.a.

   (b) In each of the above, how many thousands of dollars face value can be purchased for $100,000? How much of the $100,000 is left over?

3. Show $P = W + \frac{r}{i}(V - W)$, where $W = V(1+i)^{-n}$. (Makeham's formula).

4. Use Makeham's formula to find the price of the following bonds:

   (a) 2 years to maturity, interest at 8% p.a. to yield $12 \frac{1}{2}$ % p.a.

   (b) 10 years to maturity, interest at $13 \frac{3}{4}$% p.a. to yield 12% p.a.

   (c) 20 years to maturity, interest at $11 \frac{3}{4}$% p.a. to yield 13% p.a.

5. Consider a bond of face value $1,000 that pays 10% semi-annually, for a period of 10 years. You are willing to buy it if you can get a yield of 11%.

   (a) Let P_1 be the price you are willing to pay after the first interest payment.

   Let P_2 be the price you are willing to pay after the second interest payment.

   Let P_t be the price you are willing to pay after the tth interest payment.

   Show $P_{t+1} = P_t(1+i) - Vr$ and calculate P_1, P_2, P_3, P_4.

   (b) For $0 < k < 1$, let P_{t+k} represent the price you are willing to pay for the bond between the tth and $(t+1)^{st}$ interest payment. Since the price on this interim date is the price on the preceding payment date accumulated with interest at the yield rate for the fractional period, we must have

$$P_{t+k} = P_t(1+i)^k .$$

   Calculate $P_{1.5}$, $P_{1.75}$, $P_{1.90}$, $P_{2.5}$, $P_{3.5}$.

(c) Show $\lim\limits_{k\to 1^-} P_{t+k} = P_{t+1} + Vr$.

Is the function P_x continuous at $t+1$? (Interpret your answer.) Sketch a graph of P_x.

(d) In practice, the expression for P_{t+k} from part (b) is rarely used. Instead a linear approximation is used between the price at the beginning of the interval, P_t, and the price at the end of the interval just before the interest payment is made, $P_{t+1} + Vr$. Find the linear equation through the points (t, P_t) and $(t+1, P_{t+1} + Vr)$ and use it to show

$$P_{t+k} \simeq (1-k)P_t + kP_{t+1} + kVr, \qquad \text{for} \qquad 0 < k < 1.$$

Approximate $P_{1.5}$, $P_{1.75}$, $P_{1.90}$, $P_{2.5}$, $P_{3.5}$ and compare the exact prices from part (b).

(e) In the formula in part (d), interpret the quantity kVr.

6. Find the yield of the following bonds as of September 15, 1980.

   (a) ABC $10\frac{1}{4}$% maturing March 15, 2002, price \$87.26.

   (b) BC Hydro 10% maturing September 15, 2000, price \$71.25.

   (c) Canada $8\frac{1}{2}$% maturing March 15, 1995, price \$92.00.

   (d) Shell $9\frac{3}{8}$% maturing March 15, 2003, price \$71.75.

   (e) Canada 8% maturing September 15, 1985, price \$104.55.

7. (Methods to calculate i.)

   (a) Using $P = V(1+i)^{-n} + rVa_{\overline{n}|i}$, show

   $$i = r - \frac{P - V}{Va_{\overline{n}|i}}.$$

   This formula leads to one method for calculating i: successive approximations. [Guess a reasonable value for i and substitute it in the right side. Simplifying the right side will give you a new value of i which is closer to the actual value. Use this new value to substitute into the right side and repeat the procedure. Repeat until i shows very little change.] Try this method for the bond in 6(d). As a first guess take $i = .06$.

   (b) Another method approximates $\frac{1}{a_{\overline{n}|i}}$ by

   $$\frac{1}{n}\left[1 + \frac{n+1}{2}\cdot i + \frac{n^2-1}{12}\cdot i^2 + \cdots\right].$$

   If we ignore second and higher powers of i, this gives

   $$\frac{1}{a_{\overline{n}|i}} \simeq \frac{1}{n}\left[1 + \frac{n+1}{2}\cdot i\right].$$

5

Using this approximation and the formula from part (a), show

$$i \simeq \frac{r - \frac{P-V}{nV}}{1 + \frac{n+1}{2n} \cdot \frac{P-V}{V}} \; .$$

Calculate the yields of the bonds in question 1 with this formula.

(c) Finally, in the formula from part (b) replace

$$\frac{n+1}{2n} \qquad \text{by} \qquad \frac{1}{2}, \qquad \text{giving} \qquad i \simeq \frac{r - \frac{P-V}{nV}}{1 + \frac{P-V}{2V}} \; .$$

This is the so-called "bond salesmen's method". Use it to calculate the yields of the bonds in question 1.

(d) Comment on the relative accuracy of these three methods.

8. Mr. A. offers to liquidate a debt of \$18,000 with you by either assigning to you a mortgage described below or transferring ownership in a bond as described below. If you have equal confidence in these securities and can presently get 13% compounded semi-annually for like securities, which offer should you choose?

The Mortgage:	pays \$240 a month for the next 11 years.
The Bond:	face value of \$20,000 with a coupon rate of $10\frac{1}{2}$ % maturing in 20 years.

9. (a) Let P be the price of a bond. Show

$$\lim_{n \to \infty} P = \frac{rV}{i}$$

where n is the number of interest payments until the bond matures, r is the coupon rate per interest payment, V is the face value of the bond, and i is the yield rate per interest payment.

(b) The above result leads to an approximation for the yield, namely $i \simeq \frac{rV}{P}$. Use this formula to approximate the yield of a bond quoted (on June 1, 1981) as:

Canada 13%, June 1, 2001, \$79.50

Is your approximation for the yield too high or too low?

6