



Harmonic measure and winding of random conformal paths: A Coulomb gas perspective

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Abstract

We consider random conformally invariant paths in the complex plane (SLEs). Using the Coulomb gas method in conformal field theory, we rederive the mixed multifractal exponents associated with both the harmonic measure and winding (rotation or monodromy) near such critical curves, previously obtained by quantum gravity methods. The results also extend to the general cases of harmonic measure moments and winding of multiple paths in a star configuration.

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1. Introduction

1.1. Historical perspective

The subject of conformally invariant (CI) random curves in two dimensions has seen spectacular progress in recent years thanks to the invention of the Stochastic Loewner Evolution (SLE) [1–3]. This represents the crowning achievement of studies of 2D critical systems undertaken more than thirty-years ago. The first advance came with the introduction of the so-called Coulomb-gas (CG) formalism. The critical properties of fundamental two-dimensional statistical

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models, like the $O(N)$ and Potts models, then found an analytic description within that formalism, which led to a number of exact results [4,5].

This was soon followed by the conformal invariance breakthrough that occurred in 1984 with the celebrated BPZ article [6]. It was followed by many studies in conformal field theory (CFT), which became an essential source of applications to 2D statistical mechanics [7,8]. That finally caught the attention of mathematicians [9], first through the peculiar cases of Cardy's formula for crossing probabilities in percolation [10] and of intersection properties of planar Brownian paths [11], resulting several years later in the advent of the SLE era [12–14].

Meanwhile, the relationship between the CG and CFT approaches was brought to light when the Coulomb gas representation appeared as an explicit model for a continuum of two-dimensional CFTs [15,16]. A Gaussian free field theory in a given domain, modified by a background charge $2\alpha_0$ (the “charge at infinity”) that couples the field to the domain or boundary curvature, provides a concrete representation of abstract conformal field theories with central charge $c = 1 - 24\alpha_0^2 \leq 1$ [16]. Many of the critical properties of statistical models could then be obtained from the fusion of these approaches.

Another advance came in 1988 with the intrusion into statistical mechanics of two-dimensional quantum gravity (QG) [17]. The so-called KPZ relation between conformal weights in presence of a fluctuating metric and those in the Euclidean complex plane [17,18] could then be checked by explicit calculations [19–21]. Though it is not the subject of this article, one cannot avoid mentioning in passing the important body of knowledge accumulated since then in the related field of random matrix theory.

A decade later, just before the advent of SLE, it became clear, first from the reinterpretation of an independent rigorous study in probability theory on intersection properties of planar Brownian paths [22], that an underlying quantum gravity structure played a unifying role in conformal random geometry in the complex plane [23]. In particular, revisiting QG allowed the prediction of fine geometrical, i.e., multifractal, properties of random critical curves.

Those concerned the multifractal spectra associated with the moments of the harmonic measure [24], i.e., the electrostatic potential, near CI curves, which could be derived exactly within that approach for any value of the central charge c [25]. A generalization concerned the peculiar *mixed* multifractal spectrum that describes both the harmonic measure moments and the indefinite winding or rotation (i.e., as logarithmic spirals) [26] of a CI curve about any of its points or of the Green lines of the potential [27]. Higher multifractality spectra were also introduced and calculated, concerning multiple moments of the harmonic measure and winding in various sectors of multiple random (SLE) paths in a star configuration [28,29].

All these studies resorted to quantum gravity, by using a probabilistic representation of harmonic measure moments in terms of collections of Brownian paths, and by performing a “*transmutation*” of the latter paths into multiple (mutually-avoiding) SLEs, the rules of which are established within the QG formalism. The mixed harmonic-rotational spectrum was then obtained by blending this method with an earlier Coulomb gas study of the winding angle distributions of critical curves [30].

In mathematics, the same multifractal harmonic measure or mixed spectra are the subject of present studies and can be obtained rigorously via the probabilistic SLE approach [31,32].

Recently, the same problem of the geometrical properties of critical curves was addressed in the physics literature via the Coulomb gas approach alone [33,34]. That work in particular used the statistical equivalence between correlation functions of conformal operators in the complex plane \mathbb{C} and correlation functions of a subset of these operators in presence of SLEs, i.e., in \mathbb{C} cut by the latter [35–37]. Riemann uniformizing conformal maps were then devised to unfold

the random paths onto the outside of some smooth domain such as the unit disk. Standard CFT transformation rules of primary conformal operators were then the tool of choice. They allowed in particular to recover the multifractal exponents associated with harmonic measure moments near critical curves, originally obtained from quantum gravity.

This ingenious approach thus raised the open question of how to generalize it to mixed *harmonic and winding* properties of random conformal curves, and recover the associated mixed multifractal spectra. The aim of the present article is to present such a generalization, still solely within the Coulomb gas formalism.

Not surprisingly, the main ingredient is the use of arbitrary *chiral* primary operators, instead of the more familiar spinless ones. Their associated CFT transformation rules under arbitrary conformal maps will then yield information about the rotations that occur along Green lines near conformally invariant random paths.

A technical subtlety will arise here, since one no longer uses correlation functions of operators alone to extract geometrical information about windings of random paths. To study such windings, it is convenient to resort to *products of vertex operators*, without statistical averaging, and study their *monodromy* properties at short distance. These encode the asymptotic geometrical rotation properties of the random paths and can be analyzed via operator product expansions, as will be shown below.

1.2. Harmonic measure and winding moments

Definitions

Let us consider a random conformal path \mathcal{S} , such as the trace of a Stochastic Loewner Evolution (SLE), or a critical scaling curve in a two-dimensional critical statistical system (Fig. 1). Consider a point $0 \in \mathcal{S}$ and a point $z \notin \mathcal{S}$. Define $\tilde{\omega}(0, z)$ as the harmonic measure of the ball $B_z(0)$ of radius $|z|$, centered at 0. This harmonic measure is the probability that a Brownian motion, started “at infinity”, first hits \mathcal{S} inside the ball $B_z(0)$. Alternately, one can consider an electrostatic potential, with Dirichlet boundary condition 0 on the equipotential \mathcal{S} and value 1 on a large circle “at infinity”. The harmonic measure $\tilde{\omega}(0, z)$ is then the electric charge induced on $\mathcal{S} \cap B_z(0)$. Let $\vartheta(z)$ be the (possibly indefinite) rotation angle of the Green line (i.e., electrostatic field line) passing through point z , when z tends to 0 while avoiding \mathcal{S} ; equivalently, this is the rotation angle of the equipotential passing through z (i.e., path \mathcal{S} for $z \rightarrow 0$).

We shall evaluate the (typical) asymptotic behavior for $z \rightarrow 0$ of the *mixed moment*,

$$\tilde{\omega}^h(z) \exp(-p\vartheta(z)) \sim |z|^{\hat{x}(h,p)}, \quad (1)$$

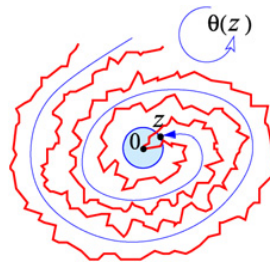


Fig. 1. A random conformal path \mathcal{S} , equipped with a ball $B_z(0)$ centered at point $0 \in \mathcal{S}$ and passing through point $z \notin \mathcal{S}$. The winding angle $\vartheta(z)$ essentially measures the (possibly multiple or indefinite) rotation of the path \mathcal{S} up to point z , when $z \rightarrow 0$.

also expected when averaging over configurations of the random path. The critical exponent $\hat{x}(h, p)$ depends on the two arbitrary parameters h and p explicitly [27].

It is also interesting to consider in general multiple SLE paths with a n -arm star topology, and to define in a similar way the multiple mixed moments [29]:

$$\prod_{i=1}^n \tilde{\omega}_i^{h_i}(z) \times \exp(-p\vartheta(z)) \sim |z|^{\hat{x}(\{h_i\}; p)}. \quad (2)$$

Here $\tilde{\omega}_i(z)$ is the harmonic measure in the i th sector of the star, within the ball $B_z(0)$ of radius $|z|$ centered at apex 0, while $\vartheta(z)$ is the common monodromy angle describing the rotation for $z \rightarrow 0$ of the Green lines between the arms of the random conformal star. The previous single path case can be seen as a two-star configuration with $\tilde{\omega}_{i=1,2}(z)$ the harmonic measures of the two sectors cut by the path inside the ball $B_z(0)$.

Here again, the critical exponent $\hat{x}(\{h_i\}, p)$ depends explicitly on all arbitrary exponents h_i and rotation parameter p [29], as we shall see in Section 3.

Riemann map

A known way of computing these moments is to consider the conformal map $w(z)$ which transforms the exterior of the random path \mathcal{S} into the exterior of the unit disk \mathbb{D} . The derivative of this map, $w'(z)$, encodes all the relevant geometrical information. For $z \rightarrow 0$, one has the equivalence for the harmonic measure $\tilde{\omega}(z) \sim |zw'(z)|$, while the winding angle is given asymptotically by $\vartheta(z) \simeq -\arg w'(z)$. Thus the mixed moments (1) above can be studied as well via the moments associated with the derivative of the conformal map

$$|w'(z)|^h \exp[p \arg w'(z)] \sim |z|^{x(h, p)}, \quad (3)$$

with a now obvious shift of the critical exponent $x(h, p) := \hat{x}(h, p) - h$.

In the case of a n -arm star, one takes a set of points z_i each in a separate sector $i = 1, \dots, n$, and with distances $|z_i| \sim |z|$ all scaling in the same way. Then the windings in each sector are asymptotically equivalent so that for all i , $\arg w'(z_i) \simeq -\vartheta(z)$, whence

$$\begin{aligned} \prod_{i=1}^n |w'(z_i)|^{h_i} \exp[p_i \arg w'(z_i)] &\sim \prod_{i=1}^n |w'(z_i)|^{h_i} \times \exp[-p\vartheta(z)] \\ &\sim |z|^{x(\{h_i\}; p)}, \end{aligned} \quad (4)$$

where $p := \sum_{i=1}^n p_i$ and $x(\{h_i\}; p) := \hat{x}(\{h_i\}; p) - \sum_i h_i$.

To keep the formalism and technical notations to a minimum, and to avoid confusing the reader, we have simply adopted the notations of previous work by the Chicago group [34], of which the present study can be considered as a generalization. The reader is thus referred to their article which contains some relevant introductory details. The connection to our previous results and notation more familiar in quantum gravity will be seen at the end of this article.

2. Derivation

2.1. Star and operator products

Vertex functions

In complex Gaussian free field theory [38,39], the basic objects considered here are “operator products”, i.e., products of so-called vertex functions:

$$\mathbb{P} := \left[\prod_{i=1}^n \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i) \right]_{\mathbb{C}}, \quad (5)$$

where each vertex function or “operator” $\mathcal{O}_{\alpha, \bar{\alpha}}(z, \bar{z})$ is formally made of two respectively holomorphic and anti-holomorphic components

$$\begin{aligned} \mathcal{O}_{\alpha, \bar{\alpha}}(z, \bar{z}) &= V^{\alpha}(z) \times \bar{V}^{\bar{\alpha}}(\bar{z}), \\ V^{\alpha}(z) &:= e^{i\sqrt{2}\alpha\phi(z)}, \quad \bar{V}^{\bar{\alpha}}(\bar{z}) := e^{i\sqrt{2}\bar{\alpha}\bar{\phi}(\bar{z})}, \end{aligned} \quad (6)$$

with Gaussian free-field correlators:

$$\begin{aligned} \langle \phi(z)\phi(z') \rangle &= -\log(z - z'), & \langle \bar{\phi}(\bar{z})\bar{\phi}(\bar{z}') \rangle &= -\log(\bar{z} - \bar{z}'), \\ \langle \phi(z)\bar{\phi}(\bar{z}') \rangle &= 0. \end{aligned} \quad (7)$$

The holomorphic and the anti-holomorphic weights of the vertex operators are found in a standard way by applying the stress–energy tensor to them:

$$h = h_{\alpha} = \alpha(\alpha - 2\alpha_0), \quad \bar{h} = h_{\bar{\alpha}} = \bar{\alpha}(\bar{\alpha} - 2\alpha_0), \quad (8)$$

where $2\alpha_0$ is the background charge acting in the Coulomb gas representation of the Gaussian free field theory. A given weight h thus corresponds to two possible charges:

$$\alpha_h = \alpha_0 \pm \sqrt{\alpha_0^2 + h}, \quad (9)$$

an equation which applies separately to holomorphic and anti-holomorphic components. A vertex operator is spinless (meaning that $h = \bar{h}$) if either $\bar{\alpha} = \alpha$ or $\bar{\alpha} = 2\alpha_0 - \alpha$. In the sequel, since we are interested in the rotation of conformally invariant curves, the main tool is the consideration of *chiral* operators, as opposed to the usual case of spinless ones. Therefore we explicitly consider in (5) operators $\mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}$ with different conformal weights:

$$h'_i := h_{\alpha'_i} \neq h_{\bar{\alpha}'_i} =: \bar{h}'_i.$$

Star operator product

To study the harmonic or rotation properties of a star configuration of n critical curves, i.e., SLEs, the main mathematical object is the *star operator product* [34]

$$\mathbb{P}_n(\{z_i, \bar{z}_i\}) := \left[\Psi_{0, n/2}(0) \prod_{i=1}^n \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i) \right]_{\mathbb{C}}, \quad (10)$$

where the operator $\Psi_{0, n/2}(0)$ in (10) is, in the conformal field theory parlance, the operator corresponding to the existence of n critical curves originating at point $z = 0$ in the plane. It represents

the seed of a star configuration of n such curves, i.e., SLEs. The *test operators* $\mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}$ are a priori chiral, so that $\alpha'_i \neq \bar{\alpha}'_i$ (or $\neq 2\alpha_0 - \bar{\alpha}'_i$).

We shall need the standard CG notations for holomorphic charges [39]:

$$\alpha_{r,s} := \alpha_0 - \frac{1}{2}(r\alpha_+ + s\alpha_-) = \frac{1}{2}(1-r)\alpha_+ + \frac{1}{2}(1-s)\alpha_-, \quad (11)$$

where the basic charges are given in terms of the SLE parameter $\kappa \in [0, 8]$ by [34]

$$\alpha_+ = \frac{\sqrt{\kappa}}{2}, \quad \alpha_- = -\frac{2}{\sqrt{\kappa}}, \quad (12)$$

$$2\alpha_0 = \alpha_+ + \alpha_- = \frac{\sqrt{\kappa}}{2} - \frac{2}{\sqrt{\kappa}}. \quad (13)$$

The charges α_{\pm} satisfy the simple relations

$$\alpha_{\pm} = \alpha_0 \pm \sqrt{\alpha_0^2 + 1},$$

thus both correspond to a conformal weight $h_{\alpha_{\pm}} = 1$, leading to the possibility of using them as fundamental “screening charges” to build up the algebra of screening operators [16].

The central charge of the CFT associated with the Gaussian free field theory, modified *à la* Feigin–Fuchs and Dotsenko–Fateev by the background charge $2\alpha_0$, is given by

$$c = 1 - 24\alpha_0^2 = 1 - \frac{3}{2} \frac{(\kappa - 4)^2}{\kappa}, \quad (14)$$

while the conformal weights corresponding to charges $\alpha_{r,s}$ are

$$h_{r,s} := \alpha_{r,s}(\alpha_{r,s} - 2\alpha_0) = \frac{(r\kappa - 4s)^2 - (\kappa - 4)^2}{16\kappa}. \quad (15)$$

In this representation, the holomorphic (or anti-holomorphic) charge of the bulk curve-creating operator $\Psi_{0,n/2}$ can be taken to be $\alpha_{0,n/2} = \alpha_0 - \frac{n}{4}\alpha_-$ (with conjugate $\alpha_{0,-n/2}$), and corresponds in the Coulomb gas formalism to a combination of electric and magnetic charges [34,40]. The corresponding operator is *spinless* with conformal weight [40]

$$h_{0,n/2} = \frac{4n^2 - (\kappa - 4)^2}{16\kappa}. \quad (16)$$

Notice that operator products such as (5) or (10) are understood as objects to be inserted in correlation functions, in the same way as the formal holomorphic vs anti-holomorphic factorization (6) is meaningful only in such correlations (6). Now, in the Coulomb gas formulation of conformal field theory, correlation functions are to be evaluated for a set of charges which overall respect electroneutrality, so that the sum of the latter always equals $-2\alpha_0$, to compensate for the existence of the background charge $2\alpha_0$. So ultimately, the set of charges introduced by the operator product \mathbb{P}_n near the origin, namely the set $\{\alpha_{0,n/2}, (\alpha'_i, \bar{\alpha}'_i), i = 1, \dots, n\}$ has to be compensated by other charges located at the observation points of the other remaining vertex functions.

When dealing with “conformal blocks”, namely treating separately the holomorphic and anti-holomorphic parts in correlation functions, electroneutrality should apply separately to both sectors. Furthermore, if holomorphic (or anti-) electroneutrality does not apply in the arguments of the correlation functions of the original vertex operators, it is known that supplementary integral *screening operators*, with screening charges α_+ and α_- (12), can be introduced to extend the domain of definition of the theory.

This caveat here is to explain why, when dealing with local operator products, such as that (5) or (10), one does not have to enforce Coulomb gas neutrality rules locally, but only globally. We are here actually concerned only with the short distance and monodromy properties of the star operator product (10) near the origin, which result from the analysis of the short-distance expansion of chiral operators $\mathcal{O}_{\alpha', \bar{\alpha}'}$ in presence of the spinless curve-creating one $\Psi_{0,n/2}$, and the CG formalism will be kept at a minimum. It would be interesting to make a more complete study of the algebra of holomorphic operators, including screening ones, for the present problem and in the context of SLE theory [16,41].

2.2. Star operator product expansion

OPE for vertex functions

Let us first consider an arbitrary product of vertex functions in the complex plane:

$$\mathbb{P} = \prod_{i \in \mathcal{I}} [\mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i)]_{\mathbb{C}} = \prod_{i \in \mathcal{I}} [V_{\alpha'_i}(z_i) \times \bar{V}_{\bar{\alpha}'_i}(\bar{z}_i)]_{\mathbb{C}}, \quad (17)$$

where we have made explicit the formal factorization into holomorphic and anti-holomorphic vertex operators.

When a subset \mathcal{P} of \mathcal{I} of points $i \in \mathcal{P}$ is *contracted* towards a common point, like the origin here, i.e., $\forall i \in \mathcal{P}, z_i \rightarrow 0$, it is well known that the limiting object is given by the so-called “operator product expansion” (OPE), with coefficients that are singular functions of the set of points \mathcal{P} .

In a Gaussian free-field theory, and for a product of vertex functions, i.e., complex exponentials of the field, these coefficients are precisely given by the average correlators of the set of vertex functions located at the contracted set \mathcal{P} . The simplest and most convenient way to express this result is probably to use the *normal ordering* of operators $:(\cdots):$ such that [42,43]

$$\prod_{i \in \mathcal{P}} [\mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i)]_{\mathbb{C}} = \left\langle \prod_{i \in \mathcal{P}} [\mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i)] \right\rangle_{\mathbb{C}} : \prod_{i \in \mathcal{P}} [\mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i)]_{\mathbb{C}} :. \quad (18)$$

In this notation, the vertex operators appearing inside $:(\cdots):$, when inserted in any correlation function, are to be Wick-contracted only with operators located *outside* of the contracting set \mathcal{P} .

In the complex plane, because of the form (7) of Gaussian averages (or “Wick contractions”, or also “free-field propagators”), the holomorphic and anti-holomorphic sector contributions to a correlation function decouple, hence factorize:

$$\begin{aligned} \left\langle \prod_{i \in \mathcal{P}} \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i) \right\rangle_{\mathbb{C}} &= \left\langle \prod_{i \in \mathcal{P}} [V_{\alpha'_i}(z_i) \bar{V}_{\bar{\alpha}'_i}(\bar{z}_i)] \right\rangle_{\mathbb{C}} \\ &= \prod_{i, j \in \mathcal{P}, i < j} (z_i - z_j)^{2\alpha'_i \alpha'_j} (\bar{z}_i - \bar{z}_j)^{2\bar{\alpha}'_i \bar{\alpha}'_j}. \end{aligned} \quad (19)$$

This in turn gives the explicit form of the OPE (18).

Star OPE

Applying the above OPE result to the star-operator product

$$\mathbb{P}_n(\{z_i, \bar{z}_i\}) := \left[\Psi_{0,n/2}(0) \prod_{i=1}^n \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i) \right]_{\mathbb{C}}, \quad (20)$$

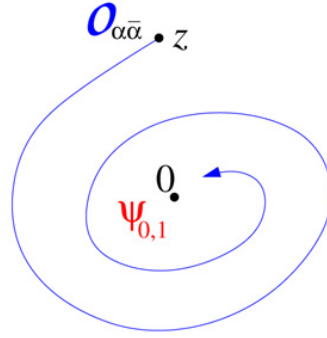


Fig. 2. Curve-creating operator $\Psi_{0,1}(0)$ and chiral test operator $O_{\alpha,\bar{\alpha}}(z, \bar{z})$, with monodromy of z around base point 0.

when the set of points $z_i, i = 1, \dots, n$ all contract to the origin $z = 0$, yields

$$\mathbb{P}_n(\{z_i, \bar{z}_i\}) = \prod_i z_i^{2\alpha_{0,n}/2\alpha'_i} \bar{z}_i^{2\alpha_{0,n}/2\bar{\alpha}'_i} \prod_{i < j} (z_i - z_j)^{2\alpha'_i\alpha'_j} (\bar{z}_i - \bar{z}_j)^{2\bar{\alpha}'_i\bar{\alpha}'_j} \times : \mathbb{P}_n(\{z_i, \bar{z}_i\}) :, \quad (21)$$

a result that will be our main tool in the analysis of the multifractal properties of a set of critical paths in a star configuration.

2.3. Monodromy and harmonic measure of a single path

Original operator product

Let us consider the standard case, where the star reduces to a single random curve passing through the origin $z = 0$, with $n = 2$ arms, i.e., a set of two semi-infinite curves arising at the origin. The associated operator is $\Psi_{0,1}$ with holomorphic charge $\alpha_{0,1} = \alpha_+/2$.

Let us place first one arbitrary “test” operator $\mathcal{O}_{\alpha',\bar{\alpha}'}(z, \bar{z})$ near the origin, so as to “measure” the harmonic measure moments and rotation on one side of the critical curve (Fig. 2). The case of two test operators on both sides of the path will be treated later as the peculiar $n = 2$ case of the general n -star geometry. In the present geometrical situation, the relevant operator product is

$$\mathbb{P}(z, \bar{z}) := [\Psi_{0,1}(0) \mathcal{O}_{\alpha',\bar{\alpha}'}(z, \bar{z})]_{\mathbb{C}}, \quad (22)$$

and we are interested in its rotation, i.e., monodromy properties at short-distance (Fig. 2). Its short-distance behavior when $z \rightarrow 0$ is a simple case of (21)

$$\mathbb{P}(z, \bar{z}) = z^{2\alpha_{0,1}\alpha'} \bar{z}^{2\alpha_{0,1}\bar{\alpha}'} : \mathbb{P}(z, \bar{z}) :, \quad (23)$$

such that the SDE coefficient (23) is explicitly

$$\mathbb{P}(z, \bar{z}) \sim z^{\alpha+\alpha'} \bar{z}^{\alpha+\bar{\alpha}'} = (z\bar{z})^{\alpha+(\alpha'+\bar{\alpha}')/2} \left(\frac{z}{\bar{z}}\right)^{\alpha+(\alpha'-\bar{\alpha}')/2}. \quad (24)$$

Operator in path geometry

One first writes the identity *in law* of the operator product \mathbb{P} (22) in \mathbb{C} , of the path creating vertex operator $\Psi_{0,1}$ by the test operator, to the same test operator but now in presence of the random path \mathcal{S} originally represented by $\Psi_{0,1}$, hence to the same test operator in the complex plane $\mathbb{C} \setminus \mathcal{S}$ slit by the curve (Fig. 3):

$$\mathbb{P}(z, \bar{z}) := [\Psi_{0,1}(0) \mathcal{O}_{\alpha',\bar{\alpha}'}(z, \bar{z})]_{\mathbb{C}} \stackrel{(\text{in law})}{=} [\mathcal{O}_{\alpha',\bar{\alpha}'}(z, \bar{z})]_{\mathbb{C} \setminus \mathcal{S}}. \quad (25)$$

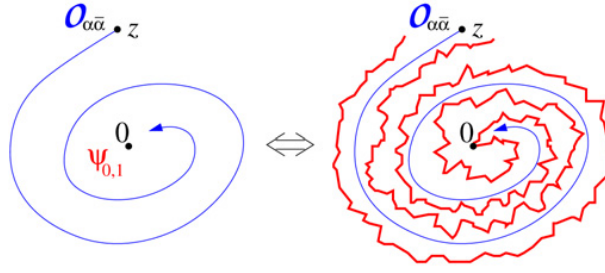


Fig. 3. Illustration of the statistical identity (25) of the product of operators $\psi_{0,1}$ and $\mathcal{O}_{\alpha,\bar{\alpha}}$ with the latter one put in presence of the stochastic path.

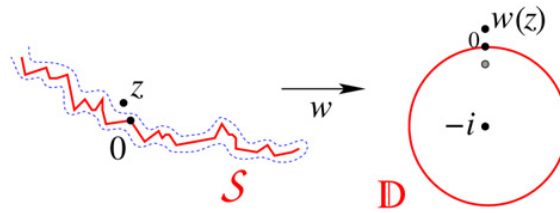


Fig. 4. Conformal map of the complement $\mathbb{C} \setminus \mathcal{S}$ of the random path \mathcal{S} in \mathbb{C} to the exterior of the unit disk \mathbb{D} . A mirror image of $w(z)$ by inversion with respect to the unit circle $\partial\mathbb{D}$ appears underneath the boundary.

This identity in law means that averaging within correlation functions the left-hand side over the complex Gaussian free field (GFF), or the right-hand side over the GFF in presence of \mathcal{S} and over the configurations of the random path \mathcal{S} yield the same result [34,35]. [As discussed in [34], the precise boundary conditions on \mathcal{S} for the geometrical random fields associated with the complex GFF may depend on the phase of the critical system (“dilute” for simple SLE paths, $\kappa \leq 4$, or “dense” for $\kappa > 4$), and are not specified here. In the holomorphic formalism, analytic continuation in κ allows one to pass from one phase to the other.]

Conformal map

The complex plane slit by \mathcal{S} has the topology of the disk. One introduces the Riemann uniformizing map $z \rightarrow w(z)$ that opens the slit \mathcal{S} into the unit disk \mathbb{D} centered at $-i$, so that $w(0) = 0$ (Fig. 4) [33,34]. This map naturally depends on the random path \mathcal{S} . Under this conformal map, vertex operators transform like primary fields, whence [39]:

$$[\mathcal{O}_{\alpha',\bar{\alpha}'}(z,\bar{z})]_{\mathbb{C}\setminus\mathcal{S}} = (w'(z))^{h'} (\overline{w'(z)})^{\bar{h}'} [\mathcal{O}_{\alpha',\bar{\alpha}'}(w(z),\overline{w(z)})]_{\mathbb{C}\setminus\mathbb{D}}, \quad (26)$$

where the holomorphic and anti-holomorphic weights of operator $\mathcal{O}_{\alpha',\bar{\alpha}'}$ are respectively

$$h' = h_{\alpha'} = \alpha'(\alpha' - 2\alpha_0), \quad \bar{h}' = h_{\bar{\alpha}'} = \bar{\alpha}'(\bar{\alpha}' - 2\alpha_0), \quad (27)$$

such that

$$\alpha' = \alpha_{h'} := \alpha_0 \pm \sqrt{\alpha_0^2 + h'}.$$

Associating together the holomorphic and anti-holomorphic weights h' and \bar{h}' , we can write identically

$$[\mathcal{O}_{\alpha',\bar{\alpha}'}(z,\bar{z})]_{\mathbb{C}\setminus\mathcal{S}} = |w'(z)|^{h'+\bar{h}'} \left(\frac{dw}{d\bar{w}}(z) \right)^{(h'-\bar{h}')/2} [\mathcal{O}_{\alpha',\bar{\alpha}'}(w(z),\overline{w(z)})]_{\mathbb{C}\setminus\mathbb{D}}, \quad (28)$$

where

$$\frac{dw}{d\bar{w}}(z) := \frac{w'(z)}{\overline{w'(z)}}.$$

The final disk configuration in $[\mathcal{O}_{\alpha', \bar{\alpha}'}(w(z), \overline{w(z)})]_{\mathbb{C} \setminus \mathbb{D}}$ is a *boundary* configuration [41], where an image charge appears at the conformal image of point $w(z)$ with respect to the disk boundary, namely its image by inversion with respect to the unit circle. In the limit $z \rightarrow 0$, since $w(0) = 0$, the latter image coincides with the image with respect to the tangent line, i.e., the complex conjugate $\overline{w(z)}$, so the same notation is kept here for simplicity. The vertex operator with anti-holomorphic charge $\bar{\alpha}'$ in $[\mathcal{O}_{\alpha', \bar{\alpha}'}]_{\mathbb{C} \setminus \mathbb{D}}$ then becomes an *holomorphic* vertex function of charge $\bar{\alpha}'$ taken at image point $\overline{w(z)}$, so that the holomorphic and anti-holomorphic sectors get coupled [39].

SDE in the w plane

When $z \rightarrow 0$ in the original domain, $w(z) \rightarrow 0$ as well as $\overline{w(z)} \rightarrow 0$, so the two image points pinch the unit circle at the origin in the w plane. The coefficient in the short-distance expansion (SDE) of the right-hand side operator in Eq. (28) is given by the Gaussian averaged correlation:

$$\langle \mathcal{O}_{\alpha', \bar{\alpha}'}(w(z), \overline{w(z)}) \rangle_{\mathbb{C} \setminus \mathbb{D}} = (w(z) - \overline{w(z)})^{2\alpha' \bar{\alpha}'}.$$

Then the short-distance expansion of Eq. (26) is

$$[\mathcal{O}_{\alpha', \bar{\alpha}'}(z, \bar{z})]_{\mathbb{C} \setminus \mathcal{S}} \sim |w'(z)|^{h' + \bar{h}'} \left(\frac{dw}{d\bar{w}}(z) \right)^{(h' - \bar{h}')/2} (w(z) - \overline{w(z)})^{2\alpha' \bar{\alpha}'}.$$
 (29)

After uniformization to the w plane, since the two image points $w(z)$ and $\overline{w(z)}$ pinch the unit circle while staying either in the exterior or the interior of the disk, they cannot wind about the origin $w(0) = 0$ indefinitely for $|z| \rightarrow 0$, so $\arg w(z)$ remains *bounded*, e.g.,

$$\arg w(z) \in [-\pi, +\pi].$$
 (30)

We conclude that $w(z) - \overline{w(z)} \sim |z| |w'(z)|$, up to a *non-winding* (or non-monodromic) complex phase factor. This finally gives the SDE (29)

$$[\mathcal{O}_{\alpha', \bar{\alpha}'}(z, \bar{z})]_{\mathbb{C} \setminus \mathcal{S}} \sim |z|^{2\alpha' \bar{\alpha}'} |w'(z)|^h \left(\frac{dw}{d\bar{w}}(z) \right)^{(h' - \bar{h}')/2},$$
 (31)

where the overall (harmonic measure) derivative exponent h is defined as

$$h := h' + \bar{h}' + 2\alpha' \bar{\alpha}'.$$
 (32)

Owing to (27), it is also

$$h = (\alpha' + \bar{\alpha}')(\alpha' + \bar{\alpha}' - 2\alpha_0).$$
 (33)

It is the weight of an operator whose conformal charge $\alpha_h = \alpha_0 \pm \sqrt{\alpha_0^2 + h}$ is just $\alpha' + \bar{\alpha}'$ (or its conjugate):

$$\alpha' + \bar{\alpha}' = \alpha_h = \alpha_0 \pm \sqrt{\alpha_0^2 + h}.$$
 (34)

Notice also that

$$h' - \bar{h}' = (\alpha' - \bar{\alpha}')(\alpha' + \bar{\alpha}' - 2\alpha_0).$$
 (35)

Winding and uniformizing map

In the z plane, the curve \mathcal{S} can wind or rotate indefinitely about any of its points, e.g. about the origin 0 when $z \rightarrow 0$, so $\arg z \rightarrow \pm\infty$. We are thus especially interested in the monodromy properties of the operator $\mathcal{O}_{\alpha', \bar{\alpha}'}(z, \bar{z})$.

By analyticity of the conformal map w onto the unit disk, $w(z) \simeq w'(z)z$, so that $\arg w(z) \simeq \arg w'(z) + \arg z$. We just have seen in (30) that $\arg w(z)$ remains bounded, so that asymptotically $\arg z \sim -\arg w'(z)$ under the conformal map.

The winding angle $\vartheta(z)$ of the Green lines of the random curve, asymptotically close to $z = 0$ on the curve, is given by

$$\vartheta(z) = \arg z = -\arg w'(z) + \mathcal{O}(1). \quad (36)$$

Therefore one can also write asymptotically

$$\vartheta(z) = -\Im \log w'(z) = -\frac{1}{2i} [\log w'(z) - \log \overline{w'(z)}] = -\frac{1}{2i} \log \frac{w'(z)}{\overline{w'(z)}}, \quad (37)$$

so that the exponential winding is

$$\left(\frac{z}{\bar{z}}\right)^{1/2} = e^{i\vartheta(z)} \asymp \left(\frac{w'(z)}{\overline{w'(z)}}\right)^{-1/2} = \left(\frac{dw}{d\bar{w}}(z)\right)^{-1/2}, \quad (38)$$

where \asymp here means equality within a non-winding (non-monodromic) phase factor. We therefore arrive at the expression for the SDE (31)

$$[\mathcal{O}_{\alpha', \bar{\alpha}'}(z, \bar{z})]_{\mathbb{C} \setminus \mathcal{S}} \sim |z|^{2\alpha' \bar{\alpha}'} |w'(z)|^h e^{-i(h' - \bar{h}')\vartheta(z)}. \quad (39)$$

Mixed moments

Let us now return to the original OPE (22), (23). We can rewrite (23), (24) as a complex scaling

$$\mathbb{P}(z, \bar{z}) = [\Psi_{0,1}(0) \mathcal{O}_{\alpha', \bar{\alpha}'}(z, \bar{z})]_{\mathbb{C}} \sim |z|^{\alpha_+ (\alpha' + \bar{\alpha}')} e^{i\alpha_+ (\alpha' - \bar{\alpha}') \arg z}. \quad (40)$$

Because of the identity in law (25), identifying (40) and the short-distance expansion (39) in the transformed slit domain yields the equivalence

$$|w'(z)|^h e^{-i[h' - \bar{h}' + \alpha_+ (\alpha' - \bar{\alpha}')] \vartheta(z)} \sim |z|^{\alpha_+ (\alpha' + \bar{\alpha}') - 2\alpha' \bar{\alpha}'}. \quad (41)$$

This scaling equivalence is an SDE result, which can be interpreted as describing typical statistical behavior, also expected to hold true in a weaker form after averaging over the configurations of the stochastic path \mathcal{S} :

$$\langle |w'(z)|^h e^{-is\vartheta(z)} \rangle \sim |z|^x, \quad (42)$$

where

$$s := h' - \bar{h}' + \alpha_+ (\alpha' - \bar{\alpha}'), \quad (43)$$

$$x := \alpha_+ (\alpha' + \bar{\alpha}') - 2\alpha' \bar{\alpha}'. \quad (44)$$

Let us now express exponent $x := x(h, is)$ solely in terms of weight h and winding conjugate parameter s . Using (33), (34) and (35) we have

$$s = (\alpha' - \bar{\alpha}')(\alpha_+ + \alpha' + \bar{\alpha}' - 2\alpha_0) = (\alpha' - \bar{\alpha}')(\alpha_+ + \alpha_h - 2\alpha_0), \quad (45)$$

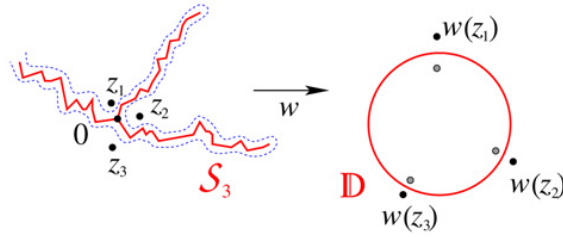


Fig. 5. Conformal map of the complement $\mathbb{C} \setminus \mathcal{S}_3$ of the random 3-star \mathcal{S}_3 in \mathbb{C} to the exterior of the unit disk \mathbb{D} . Three mirror images of points $w(z_i)$, $i = 1, 2, 3$, appear by inversion with respect to the unit circle $\partial\mathbb{D}$.

$$x = \alpha_+ \alpha_h - 2\alpha' \bar{\alpha}'. \quad (46)$$

We can further write trivially

$$4\alpha' \bar{\alpha}' = (\alpha' + \bar{\alpha}')^2 - (\alpha' - \bar{\alpha}')^2 = \alpha_h^2 - (\alpha' - \bar{\alpha}')^2,$$

to eliminate $\alpha' - \bar{\alpha}'$ in (45) and (46)

$$\alpha' - \bar{\alpha}' = \frac{p}{\alpha_+ + \alpha_h - 2\alpha_0}, \quad (47)$$

$$x(h, is) = \alpha_+ \alpha_h - \frac{1}{2} \alpha_h^2 + \frac{1}{2} \frac{s^2}{(\alpha_+ + \alpha_h - 2\alpha_0)^2}. \quad (48)$$

To get the *mixed harmonic measure-rotation exponents* one has first to *analytically continue* s : $s = -ip$ in (42) so that

$$\prec |w'(z)|^h e^{-p\vartheta(z)} \succ \sim |z|^{x(h,p)}, \quad (49)$$

with now

$$x(h, p) = \alpha_+ \alpha_h - \frac{1}{2} \alpha_h^2 - \frac{1}{2} \frac{p^2}{(\alpha_+ + \alpha_h - 2\alpha_0)^2}. \quad (50)$$

Finally, one has to choose the value of α_h that vanishes with h , namely the “dilute phase” or simple SLE path one, where $\kappa \leq 4$ and $\alpha_0 \leq 0$ (see (13))

$$\alpha_h = \alpha_0 + \sqrt{\alpha_0^2 + h}. \quad (51)$$

The exponent $x(h, p)$ above is identical to the scaling exponent obtained by quantum gravity in [27] for the h th power $\tilde{\omega}^h(z) e^{-p\vartheta(z)}$ of the harmonic measure $\tilde{\omega}$ with rotation conjugate parameter p (up to a natural shift by h , due to the local scaling $\tilde{\omega}(z) \sim |z| |w'(z)|$ for the harmonic measure $\tilde{\omega}(z)$ in a ball of radius $|z|$, cf. Eqs. (1), (3)). [See Section 3.]

2.4. Monodromy and harmonic measure of multiple paths

Operator product in star geometry

We first use the identity *in law* of the star-operator product (20) with the operator product of the test vertex operators in presence of the stochastic star \mathcal{S}_n in the complex plane:

$$\mathbb{P}_n(\{z_i, \bar{z}_i\}) \stackrel{(\text{in law})}{=} \left[\prod_i \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i) \right]_{\mathbb{C} \setminus \mathcal{S}_n}. \quad (52)$$

Conformal map

The plane slit by the star having the topology of the disk, a conformal map $w(z)$ transforms the open set $\mathbb{C} \setminus \mathcal{S}_n$ into the exterior of the unit disk \mathbb{D} [34]. The vertex operator product \mathbb{P}_n (52) in presence of \mathcal{S}_n is transformed, according to primary operator rules, as:

$$\begin{aligned} \mathbb{P}_n &= \left[\prod_i \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(z_i, \bar{z}_i) \right]_{\mathbb{C} \setminus \mathcal{S}_n} \\ &= \prod_i (w'(z_i))^{h'_i} (\overline{w'(z_i)})^{\bar{h}'_i} \left[\prod_i \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(w(z_i), \overline{w(z_i)}) \right]_{\mathbb{C} \setminus \mathbb{D}}. \end{aligned} \quad (53)$$

Associating for each i the holomorphic and anti-holomorphic weights h'_i and \bar{h}'_i , we can write it identically

$$\mathbb{P}_n = \prod_i |w'(z_i)|^{h'_i + \bar{h}'_i} \left(\frac{dw}{d\bar{w}}(z_i) \right)^{(h'_i - \bar{h}'_i)/2} \left[\prod_i \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(w(z_i), \overline{w(z_i)}) \right]_{\mathbb{C} \setminus \mathbb{D}}. \quad (54)$$

Short-distance expansion

Now let all points converge to the origin in the original z plane, $z_i \rightarrow 0, \forall i = 1, \dots, n$. As in Section 2.3 above, in the disk geometry, the operators on the right-hand side of (54) are *boundary* operators, each of them made of a pair of holomorphic vertex operators taken at $w(z_i)$ and at its inverted mirror image with respect with the unit circle (Fig. 5). When $z_i \rightarrow 0$, each image coincides with the local complex conjugate $\overline{w(z_i)}$ with respect to the local tangent to the unit circle, so we keep the same notation (a slight abuse).

As usual, the original anti-holomorphic operators with charges $\bar{\alpha}'_i$ now become holomorphic vertex operators located at points $\overline{w(z_i)}$, and get coupled to the original holomorphic vertex operators with charges α'_i located at points $w(z_i)$ [39]. Under the contraction $z_i \rightarrow 0, \forall i = 1, \dots, n$, the SDE of the transformed operator product on the right-hand side of (54) therefore scales as the Gaussian free-field average taken at all pairs of points and images [34]:

$$\begin{aligned} &\left\langle \prod_i \mathcal{O}_{\alpha'_i, \bar{\alpha}'_i}(w(z_i), \overline{w(z_i)}) \right\rangle_{\mathbb{C} \setminus \mathbb{D}} \\ &= \prod_i (w(z_i) - \overline{w(z_i)})^{2\alpha'_i \bar{\alpha}'_i} \end{aligned} \quad (55)$$

$$\times \prod_{i < j} (w(z_i) - w(z_j))^{2\alpha'_i \alpha'_j} (\overline{w(z_i)} - \overline{w(z_j)})^{2\bar{\alpha}'_i \bar{\alpha}'_j} \prod_{i \neq j} (w(z_i) - \overline{w(z_j)})^{2\alpha'_i \bar{\alpha}'_j}. \quad (56)$$

Among all these factors, the diagonal (i) terms give the short-distance behavior. The $i \neq j$ terms are finite in the disk geometry, or give subleading contributions for configurations of the star in which a pair of points (z_i, z_j) lie in the same sector, since the relative affixes $w(z_i) - w(z_j)$, $\overline{w(z_i)} - \overline{w(z_j)}$, $w(z_i) - \overline{w(z_j)}$ then tend to zero. The short-distance behavior of the operator product (54) is thus

$$\mathbb{P}_n \sim \prod_i |w'(z_i)|^{h'_i + \bar{h}'_i} \left(\frac{dw}{d\bar{w}}(z_i) \right)^{(h'_i - \bar{h}'_i)/2} (w(z_i) - \overline{w(z_i)})^{2\alpha'_i \bar{\alpha}'_i}.$$

When $z_i \rightarrow 0$, the transformed complex coordinate $w(z_i)$ and its image $\overline{w(z_i)}$ pinch the unit circle, so that $w(z_i) \rightarrow \overline{w(z_i)}$ while the winding angle $\arg w(z_i)$ stays *bounded*; whence we can

simply take: $w(z_i) - \overline{w(z_i)} \sim |z_i| |w'(z_i)|$, up to a non-monodromic complex phase factor. This gives the final short-distance behavior of (54)

$$\mathbb{P}_n \sim \prod_i |z_i|^{2\alpha'_i \bar{\alpha}'_i} |w'(z_i)|^{h'_i + \bar{h}'_i + 2\alpha'_i \bar{\alpha}'_i} \left(\frac{dw}{d\bar{w}}(z_i) \right)^{(h'_i - \bar{h}'_i)/2}. \quad (57)$$

Hence the following exponents appear for the harmonic measure derivative terms:

$$h_i := h'_i + \bar{h}'_i + 2\alpha'_i \bar{\alpha}'_i = (\alpha'_i + \bar{\alpha}'_i)(\alpha'_i + \bar{\alpha}'_i - 2\alpha_0), \quad (58)$$

which are the conformal weights of operators whose conformal charges simply result from the addition of holomorphic and anti-holomorphic charges

$$\alpha_{h_i} = \alpha_0 + \sqrt{\alpha_0^2 + h_i} = \alpha'_i + \bar{\alpha}'_i.$$

Windings and uniformizing map

Let us now return to the SDE (21) in the original z plane

$$\mathbb{P}_n(\{z_i, \bar{z}_i\}) \sim \prod_i z_i^{2\alpha_{0,n/2}\alpha'_i} \bar{z}_i^{2\alpha_{0,n/2}\bar{\alpha}'_i} \prod_{i < j} (z_i - z_j)^{2\alpha'_i \alpha'_j} (\bar{z}_i - \bar{z}_j)^{2\bar{\alpha}'_i \bar{\alpha}'_j} \quad (59)$$

and, as in Eq. (24), let us separate the complex modulus and argument parts, by rewriting it as:

$$\begin{aligned} \mathbb{P}_n(\{z_i, \bar{z}_i\}) &\sim \prod_i |z_i|^{2\alpha_{0,n/2}(\alpha'_i + \bar{\alpha}'_i)} \left(\frac{z_i}{\bar{z}_i} \right)^{\alpha_{0,n/2}(\alpha'_i - \bar{\alpha}'_i)} \\ &\times \prod_{i < j} |z_i - z_j|^{2\alpha'_i \alpha'_j + 2\bar{\alpha}'_i \bar{\alpha}'_j} \left(\frac{z_i - z_j}{\bar{z}_i - \bar{z}_j} \right)^{\alpha'_i \alpha'_j - \bar{\alpha}'_i \bar{\alpha}'_j}. \end{aligned} \quad (60)$$

We are interested in the monodromy properties of expression (60) when all points in their own sectors converge to the star apex $z = 0$ while avoiding touching the star, which can wind indefinitely about its origin. Let us then introduce a common scaling $|z|$ for all distances $|z_i|$ to the star apex, as well as a common asymptotic winding angle $\vartheta(z)$ for all arguments $\arg z_i$:

$$z_i = |z| e^{i\vartheta(z)} \zeta_i, \quad \arg z_i = \vartheta(z) + \arg \zeta_i, \quad (61)$$

where all moduli $|\zeta_i|$ and arguments $\arg \zeta_i$ remain bounded when $|z| \rightarrow 0$. Since trivially

$$(z_i / \bar{z}_i)^{1/2} = e^{i\vartheta(z)} (\zeta_i / \bar{\zeta}_i)^{1/2}, \quad (62)$$

the short-distance expansion (60) obeys the simple identity

$$\begin{aligned} \mathbb{P}_n(\{z_i, \bar{z}_i\}) &= |z|^{2\alpha_{0,n/2} \sum_i (\alpha'_i + \bar{\alpha}'_i) + \sum_{i < j} (2\alpha'_i \alpha'_j + 2\bar{\alpha}'_i \bar{\alpha}'_j)} \\ &\times e^{i\vartheta(z) [2\alpha_{0,n/2} \sum_i (\alpha'_i - \bar{\alpha}'_i) + 2 \sum_{i < j} (\alpha'_i \alpha'_j - \bar{\alpha}'_i \bar{\alpha}'_j)]} \\ &\times \mathbb{P}_n(\{\zeta_i, \bar{\zeta}_i\}). \end{aligned} \quad (63)$$

Let us now consider the geometrical setting in the transformed w -plane. Since all arguments $\arg w(z_i)$ stay bounded there, we have for $z_i \rightarrow 0$, as in Eq. (36), $\arg w'(z_i) = -\arg z_i + \mathcal{O}(1)$, so that, as in (38)

$$\left(\frac{z_i}{\bar{z}_i} \right)^{1/2} \asymp \left(\frac{dw}{d\bar{w}}(z_i) \right)^{-1/2}. \quad (64)$$

Identity (62) then yields immediately the common asymptotic behavior for all i

$$\forall i, \left(\frac{dw}{d\bar{w}}(z_i) \right)^{-1/2} \asymp e^{i\vartheta(z)}, \quad (65)$$

in terms of the unique rotation angle $\vartheta(z)$ (61), and up to non-monodromic phase factors. Applying this to the SDE result (57) gives the asymptotic formula

$$\mathbb{P}_n \sim |z|^{2\sum_i \alpha'_i \bar{\alpha}'_i} e^{-i\vartheta(z) \sum_i (h'_i - \bar{h}'_i)} \prod_i |w'(z_i)|^{h_i}. \quad (66)$$

Multiple moments and rotation

Because of the identity in law (52), we can now identify expressions (63) and (66) and get the fundamental scaling formula for the multiple harmonic measure factors and (indefinitely) rotating phase factor

$$\begin{aligned} & e^{-i\vartheta(z)[\sum_i (h'_i - \bar{h}'_i) + 2\alpha_{0,n/2} \sum_i (\alpha'_i - \bar{\alpha}'_i) + 2\sum_{i<j} (\alpha'_i \alpha'_j - \bar{\alpha}'_i \bar{\alpha}'_j)]} \times \prod_i |w'(z_i)|^{h_i} \\ & \sim |z|^{2\alpha_{0,n/2} \sum_i (\alpha'_i + \bar{\alpha}'_i) + \sum_{i<j} (2\alpha'_i \alpha'_j + 2\bar{\alpha}'_i \bar{\alpha}'_j) - \sum_i 2\alpha'_i \bar{\alpha}'_i}. \end{aligned} \quad (67)$$

We obtained this scaling relation from a short-distance operator product expansion, which yields the leading (typical) scaling behaviour of the product of harmonic measure factors and rotation phase factor. It can naturally be expected to also give the weaker result on the statistical average $\langle \dots \rangle$ of the same quantity over the configurations of the multiple random paths of star \mathcal{S}_n :

$$\langle \prod_i |w'(z_i)|^{h_i} \times e^{-i\vartheta s} \rangle \sim |z|^{x(\{h_i\}; is)}, \quad (68)$$

with the parametric representation in terms of holomorphic and anti-holomorphic charges

$$h_i := h'_i + \bar{h}'_i + 2\alpha'_i \bar{\alpha}'_i, \quad (69)$$

$$h'_i = \alpha'_i(\alpha'_i - 2\alpha_0); \quad \bar{h}'_i = \bar{\alpha}'_i(\bar{\alpha}'_i - 2\alpha_0), \quad (70)$$

$$s := \sum_i (h'_i - \bar{h}'_i) + 2\alpha_{0,n/2} \sum_i (\alpha'_i - \bar{\alpha}'_i) + 2 \sum_{i<j} (\alpha'_i \alpha'_j - \bar{\alpha}'_i \bar{\alpha}'_j), \quad (71)$$

$$x := 2\alpha_{0,n/2} \sum_i (\alpha'_i + \bar{\alpha}'_i) + \sum_{i<j} (2\alpha'_i \alpha'_j + 2\bar{\alpha}'_i \bar{\alpha}'_j) - \sum_i 2\alpha'_i \bar{\alpha}'_i. \quad (72)$$

It remains to find the explicit expression of the scaling exponent $x := x(\{h_i\}; is)$ in terms of the set of weights h_i and the Fourier variable s conjugate to rotation ϑ . Let us introduce the notations

$$h_{\alpha'} = \alpha'(\alpha' - 2\alpha_0), \quad h_{\bar{\alpha}'} = \bar{\alpha}'(\bar{\alpha}' - 2\alpha_0),$$

$$\alpha' := \sum_i \alpha'_i, \quad \bar{\alpha}' := \sum_i \bar{\alpha}'_i,$$

$$\alpha_{h_i} = \alpha'_i + \bar{\alpha}'_i,$$

$$\alpha_{\{h\}} := \alpha' + \bar{\alpha}' = \sum_i (\alpha'_i + \bar{\alpha}'_i) = \sum_i \alpha_{h_i}.$$

A little bit of algebra first shows that s (71) can be written in the compact form

$$\begin{aligned}
 s &= h_{\sum_i \alpha'_i} + 2\alpha_{0,n/2} \sum_i \alpha'_i - h_{\sum_i \bar{\alpha}'_i} - 2\alpha_{0,n/2} \sum_i \bar{\alpha}'_i \\
 &= h_{\alpha'} - h_{\bar{\alpha}'} + 2\alpha_{0,n/2}(\alpha' - \bar{\alpha}') \\
 &= (\alpha' - \bar{\alpha}')(\alpha' + \bar{\alpha}' - 2\alpha_0 + 2\alpha_{0,n/2}) \\
 &= (\alpha' - \bar{\alpha}')(\alpha_{\{h\}} - 2\alpha_0 + 2\alpha_{0,n/2}).
 \end{aligned} \tag{73}$$

One can also check that the exponent x (72) can be written in the compact form

$$\begin{aligned}
 x &= 2\alpha_{0,n/2} \sum_i \alpha_{h_i} + \left(\sum_i \alpha_{h_i} \right)^2 - \sum_i \alpha_{h_i}^2 - 2\alpha' \bar{\alpha}' \\
 &= 2\alpha_{0,n/2} \alpha_{\{h\}} + \alpha_{\{h\}}^2 - \sum_i \alpha_{h_i}^2 - 2\alpha' \bar{\alpha}'.
 \end{aligned} \tag{74}$$

Note that after this compaction of formulae, the expressions (73) and (74) are similar to expressions (45) and (46). We therefore eliminate $\alpha' - \bar{\alpha}'$ in the same way as above and arrive at a formula similar to (47) and (48)

$$\alpha' - \bar{\alpha}' = \frac{s}{\alpha_{\{h\}} - 2\alpha_0 + 2\alpha_{0,n/2}}, \tag{75}$$

$$x(\{h_i\}; is) = 2\alpha_{0,n/2} \alpha_{\{h\}} + \frac{1}{2} \alpha_{\{h\}}^2 - \sum_i \alpha_{h_i}^2 + \frac{1}{2} \frac{s^2}{(\alpha_{\{h\}} - 2\alpha_0 + 2\alpha_{0,n/2})^2}, \tag{76}$$

where we recall that $\alpha_{\{h\}} = \sum_i \alpha_{h_i}$, $\alpha_{h_i} = \alpha_0 + \sqrt{\alpha_0 + h_i}$, hence exponent x has now an explicit form in terms of the set of weights $\{h_i\}$ and of s .

It remains to analytically continue s into $s = -ip$ to get the expectation of the multiple harmonic measure moments and Laplace transform of the rotation:

$$\langle \prod_i |w'(z_i)|^{h_i} \times e^{-p\vartheta} \rangle \sim |z|^{x(\{h_i\}; p)}, \tag{77}$$

$$x(\{h_i\}; p) = 2\alpha_{0,n/2} \alpha_{\{h\}} + \frac{1}{2} \alpha_{\{h\}}^2 - \sum_i \alpha_{h_i}^2 - \frac{1}{2} \frac{p^2}{(\alpha_{\{h\}} - 2\alpha_0 + 2\alpha_{0,n/2})^2}. \tag{78}$$

The CG formula (78) for $p = 0$ coincides naturally with the result found in Ref. [34].

3. Comparison to quantum gravity results

In previous work [29] we introduced for SLE_κ the KPZ relation

$$h = \mathcal{U}_\kappa(\Delta) := \frac{1}{4} \Delta (\kappa \Delta + 4 - \kappa)$$

between conformal weights Δ in a fluctuating metric with a conformal factor given by a Gaussian free field (two-dimensional “quantum gravity” (QG)) and conformal weights h in the complex plane. Its inverse reads

$$\Delta = \mathcal{U}_\kappa^{-1}(h) = \frac{1}{2\kappa} \sqrt{16\kappa h + (\kappa - 4)^2} + \frac{1}{2} \left(1 - \frac{4}{\kappa} \right).$$

Using (13) and (51) we identify

$$\alpha_h = \frac{\sqrt{\kappa}}{2} \mathcal{U}_\kappa^{-1}(h). \quad (79)$$

The CG results above can be written as

$$\begin{aligned} x(\{h_i\}; p) &= x(\{h_i\}; 0) - \frac{\kappa}{2} \frac{p^2}{L_{\{h\}}^2}, \\ L_{\{h\}} &:= \sqrt{\kappa}(\alpha_{\{h\}} - 2\alpha_0 + 2\alpha_{0,n/2}), \\ x(\{h_i\}; 0) &= 2\alpha_{0,n/2}\alpha_{\{h\}} + \frac{1}{2}\alpha_{\{h\}}^2 - \sum_i \alpha_{h_i}^2. \end{aligned} \quad (80)$$

Using (79) we find their equivalent quantum gravity expressions [29]

$$x(\{h_i\}; p) = x(\{h_i\}; 0) - \frac{\kappa}{2} \frac{p^2}{L_{\{h\}}^2}, \quad (81)$$

$$L_{\{h\}} = \frac{\kappa}{2} \sum_i \mathcal{U}_\kappa^{-1}(h_i) + n, \quad (82)$$

$$x(\{h_i\}; 0) + \sum_i h_i = 2\mathcal{U}_\kappa \left[\frac{1}{2} \left(\frac{2}{\kappa} L_{\{h\}} + 1 - \frac{4}{\kappa} \right) \right] \quad (83)$$

$$- 2\mathcal{U}_\kappa \left[\frac{1}{2} \left(\frac{2}{\kappa} n + 1 - \frac{4}{\kappa} \right) \right]. \quad (84)$$

[As seen in (4), the term $\sum_i h_i$ simply comes from the passage from derivative moments to harmonic measure ones.]

Result (50) for the multifractal exponent $x(h, p)$ of a single-sided SLE simply corresponds to $n = 2$ and $(h_1, h_2) = (h, 0)$ in QG formulae (81)–(84).

The interpretation of these formulae is quite clear:

- Eq. (81) is the Coulomb gas formula obtained a long time ago for the exponent governing the winding angle distribution of a star made of a number $L_{\{h\}}$ of random paths (SLEs) [27, 29,30];
- $L_{\{h\}}$ (82) represents, in a star topology, the *effective number* of SLEs that are exactly equivalent through QG to a collection of h_i , $i = 1, \dots, n$, Brownian paths (which represent the set of powers h_i of the harmonic measure), in addition to the n original SLEs [28,29];
- The scaling dimension (83) is the result obtained through QG construction rules for (twice) the conformal weight in \mathbb{C} of a random star made precisely of this effective number $L_{\{h\}}$ of SLEs, while (84) is (twice) the weight of the original n -SLE star without the auxiliary harmonic Brownian paths. Eq. (83) is therefore also exactly (twice) the conformal weight $h_\alpha = \alpha(\alpha - 2\alpha_0)$ corresponding to the holomorphic charge $\alpha = \alpha_{0,L_{\{h\}}/2}$ of the curve-creating operator $\Psi_{0,L_{\{h\}}/2}$, which gives the final and rather elegant CG formula for (80):

$$x(\{h_i\}; 0) + \sum_i h_i = 2\alpha_{0,L_{\{h\}}/2}(\alpha_{0,L_{\{h\}}/2} - 2\alpha_0) - 2\alpha_{0,n/2}(\alpha_{0,n/2} - 2\alpha_0).$$

Conclusion

Results (81)–(84) can be obtained by quantum gravity construction rules almost immediately, and have a natural interpretation in that formalism. As we saw, they can also be recovered under the form (78) in the fully developed Coulomb gas and conformal field theory approach, but though the latter is quite interesting, it also appears to be significantly more cumbersome.

Its most important aspect is probably the renewed suggestion that a systematic yet rigorous representation of the stochastic properties of SLEs via a Coulomb gas driven by a Gaussian free field must exist [44], mimicking the (heuristic) physics CG approach's long-established predictive power. Let us finally mention that the mixed multifractal moments, introduced here to study the mixed harmonic measure-rotation spectrum of critical random paths, are amenable to an approach using only the Stochastic Loewner Evolution, that should allow one to establish rigorously the present results [31].

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