ON COMPUTATIONAL COMPLEXITY OF SIEGEL JULIA SETS

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ABSTRACT. It is known that some polynomial Julia sets are algorithmically impossible to draw with arbitrary magnification. On the other hand, for a large class of examples the problem of drawing a picture has polynomial complexity. In this paper we demonstrate the existence of computable quadratic Julia sets whose computational complexity is arbitrarily high.

1. FOREWORD

Let us informally say that a compact set in the plane is computable if one can program a computer to draw a picture of this set on the screen, with an arbitrary desired magnification. It was recently shown by the second and third authors, that some Julia sets are not computable [BY]. This in itself is quite surprising to dynamicists – Julia sets are among the "most drawn" objects in contemporary mathematics, and numerous algorithms exist to produce their pictures. In the cases when one has not been able to produce informative pictures (the dynamically pathological cases, like maps with a Cremer or a highly Liouville Siegel point) the feeling had been that this was due to the immense computational resources required by the known algorithms.

The next surprise came with the discovery by the authors of this paper in [BBY] that all Cremer quadratics (or more generally, rational maps without rotation domains) have computable Julia sets. The non-computable examples constructed in [BY] were Siegel quadratic polynomials, and one would expect the Cremer case to be at least as bad if not worse computationally.

The natural question to ask is then whether in those cases in which we know the Julia set is computable, but no good pictures exist, the computational complexity of such a set is indeed high. Here at least, our original intuition seems to be correct: it is shown in the present paper that there exist computable Siegel quadratic Julia sets with arbitrarily high computational complexity. An irritating possibility still remains that some Cremer Julia sets are computationally easy (and we just do not go about trying to draw them in the right way). This, however, seems unlikely. We note that the examples constructed in this paper are the first known cases of Julia sets which are not poly-time computable. The second author [Brv] and independently Rettinger [Ret] have previously shown that hyperbolic

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Julia sets are poly-time computable. It is an open question whether any polynomial Julia set with a parabolic point is not poly-time.

The structure of the paper is as follows. In §2.2 of the Introduction, having stated the principal definitions, we formulate the main result of the paper. In §2.4 we give a sketch of the argument. In §4 we formulate several technical lemmas proved in [BY]. The final §5 contains the proof of the Main Theorem.

2. Introduction

2.1. **Computability of real sets.** The reader is directed to [BY] for the discussion of the notion of computability of subsets of $\mathbb{R}^n$ as applied, in particular, to Julia sets. We recall the principal definitions here.

Denote by $\mathbb{B}$ the set of the *dyadic rationals*, i.e. rationals of the form $\frac{p}{2^m}$. Rationals in $\mathbb{B}$ can be easily represented as binary numbers. We say that $\phi : \mathbb{N} \to \mathbb{B}$ is an *oracle* for a real number x, if $|x - \phi(n)| < 2^{-n}$ for all $n \in \mathbb{N}$. In other words, ϕ provides a good dyadic approximation for x. We say that a TM M^ϕ is an *oracle machine*, if at every step of the computation M is allowed to query the value $\phi(n)$ for any n. This definition allows us to define the computability of real functions on compact sets.

Definition 2.1. We say that a function $f : [a, b] \to [c, d]$ is computable, if there exits an oracle TM $M^\phi(m)$ such that if ϕ is an oracle for $x \in [a, b]$, then on input m, M^ϕ outputs a $y \in \mathbb{B}$ such that $|y - f(x)| < 2^{-m}$.

Let $K \subset \mathbb{R}^k$ be a compact set. We say that a TM M computes the set K if it approximates K in the *Hausdorff metric*. Recall that the Hausdorff metric is a metric on compact subsets of $\mathbb{R}^n$ defined by

$$d_H(X, Y) = \inf\{\epsilon > 0 \mid X \subset U_\epsilon(Y) \text{ and } Y \subset U_\epsilon(X)\}, \tag{2.1}$$

where $U_\epsilon(S)$ is defined as the union of the set of ϵ-balls with centers in S.
We approximate C using a class $\mathcal{C}$ of sets which is dense in metric d_H among the compact sets and which has a natural correspondence to binary strings. Namely $\mathcal{C}$ is the set of finite unions of dyadic balls:

$$\mathcal{C} = \left\{ \bigcup_{i=1}^{n} \overline{B(d_i, r_i)} \mid \text{where } d_i, r_i \in \mathbb{B} \right\}.$$

Members of $\mathcal{C}$ can be encoded as binary strings in a natural way.
We now define the notion of computability of subsets of $\mathbb{R}^n$ (see [Wei], and also [RW]).

Definition 2.2. We say that a compact set $K \subset \mathbb{R}^k$ is computable, if there exists an oracle TM $M(d, n)$, where $d \in \mathbb{B}$, $n \in \mathbb{N}$ which computes a function from the family $\mathcal{F}_K$ of functions of the form

$$f(d, n) = \begin{cases} 0, & \text{if } \operatorname{dist}(d, K) > 2 \cdot 2^{-n} \\ 1, & \text{if } \operatorname{dist}(d, K) < 2^{-n} \\ 0 \text{ or } 1, & \text{otherwise} \end{cases} \tag{2.2}$$

Theorem 2.1 (see [Brv])**.** *For a compact $K \subset \mathbb{R}^k$ the following are equivalent:*

(1) K is computable as per definition 2.2,

(2) there exists a TM $M(m)$, such that on input m, $M(m)$ outputs an encoding of $C_m \in \mathcal{C}$ such that $d_H(K, C_m) < 2^{-m}$ (global computability),

(3) (in the case $k = 1, 2$) K can be drawn on a computer screen with arbitrarily good precision,

(4) the distance function $d_K(x) = \inf\{|x - y| \mid y \in K\}$ *is computable as per definition 2.1.*

In the present paper we are interested in questions concerning the computability of the Julia set $J_c = J(f_c) = J(z^2 + c)$. Since there are uncountably many possible parameter values for c, we cannot expect for each c to have a machine M such that M computes J_c (recall that there are countably many TMs). On the other hand, it is reasonable to want M to compute J_c with an oracle access to c. Define the function $J : \mathbb{C} \to K^*$ (K^* is the set of all compact subsets of $\mathbb{C}$) by $J(c) = J(f_c)$. In a complete analogy to Definition 2.1 we can define

Definition 2.3. We say that a function $\kappa : S \to K^*$ for some bounded set S is computable, if there exits an oracle TM $M^\phi(d, n)$, where ϕ is an oracle for $x \in S$, which computes a function (2.2) of the family $\mathcal{F}_{\kappa(x)}$.

Equivalently, there exists an oracle TM $M^\phi(m)$ with ϕ again representing $x \in S$ such that on input m, M^ϕ outputs a $C \in \mathcal{C}$ such that $d_H(C, \kappa(x)) < 2^{-m}$.

In the case of Julia sets:

Definition 2.4. We say that J_c is computable if the function $J : d \mapsto J_d$ is computable on the set $\{c\}$.

We have the following (see [Brv]):

Theorem 2.2. *Suppose that a TM M^ϕ computes the function J on a set $S \subset \mathbb{C}$. Then J is continuous on S in Hausdorff sense.*

The second and third authors have demonstrated in [BY]:

Theorem 2.3. *There exists a parameter value $c \in \mathbb{C}$ such that the Julia set of the quadratic polynomial $f_c(z) = z^2 + c$ is not computable.*

The quadratic polynomials in Theorem 2.3 possess Siegel disks. It was further shown by the authors of the present paper in [BBY] that the absence of rotation domains, that is either Siegel disks or Herman rings, guarantees computability of the rational Julia set. This implies, in particular, that all Cremer Julia sets are computable – this despite the fact that no informative high resolution images of such sets have ever been produced. One expects, however, that such "bad" but still computable examples have high algorithmic complexity, which makes the computational cost of producing such a picture impossibly high. We note, that the second author [Brv] and independently Rettinger [Ret] have shown:

Theorem 2.4. *Hyperbolic Julia sets are computable in polynomial time. That is, if J is the Julia set of a hyperbolic rational mapping R, then there exists a TM $M(d,n)$ which computes a function of the family (2.2) in time polynomial in the bit size of the input (d,n). It is worth noting that the same oracle TM $M^\phi(d,n)$ with the oracle representing the parameters of the rational mapping R, can be selected for all hyperbolic Julia sets. Moreover, the asymptotics of the polynomial time bound depends only on R but not on the input (d,n).*

2.2. **Statement of the Main Theorem.** On the other end of the complexity spectrum we expect to find "bad" but computable Siegel Julia sets and Cremer Julia sets. Indeed, it the present paper we show:

Theorem 2.5. *There exist quadratic Siegel Julia sets of arbitrarily high computational complexity. More precisely, for any computable increasing function $h:\mathbb{N}\to\mathbb{N}$ there exists a computable Siegel parameter value $c\in\mathbb{C}$ such that:*

- *the Julia set J_c is computable by an oracle TM;*
- *for any oracle TM $M^\phi(m)$ which computes the 2^{-m}-approximations to J_c, there exists a sequence $\{m_i\}_{i=1}^\infty$ such that M^ϕ requires the time of at least $h(m_i)$ to compute the approximation $C_{m_i}\in\mathcal{C}$.*

From this statement for global computational complexity easily follows the corresponding local statement (see [Brv] for details):

Corollary 2.6. *There exist computable parameter values c for which the Julia set J_c is computable, and the complexity of the problem of computing a function (2.2) in the family $\mathcal{F}_{J_c}$ is arbitrarily high.*

2.3. **Siegel disks of quadratic maps.** Let us discuss in more detail the occurrence of Siegel disks in the quadratic family. For a number $\theta\in[0,1)$ denote $[r_0,r_1,\ldots,r_n,\ldots]$, $r_i\in\mathbb{N}\cup\{\infty\}$ its possibly finite continued fraction expansion:

$$[r_0,r_1,\ldots,r_n,\ldots]\equiv\cfrac{1}{r_0+\cfrac{1}{r_1+\cfrac{1}{\cdots+\cfrac{1}{r_n+\cdots}}}} \tag{2.3}$$

Such an expansion is defined uniquely if and only if $\theta\notin\mathbb{Q}$. In this case, the *rational convergents* $p_n/q_n=[r_0,\ldots,r_{n-1}]$ are the closest rational approximants of θ among the numbers with denominators not exceeding q_n. In fact, setting $\lambda=e^{2\pi i\theta}$, we have

$$|\lambda^h-1|>|\lambda^{q_n}-1| \text{ for all } 0<h<q_{n+1},\ h\neq q_n.$$

The difference $|\lambda^{q_n}-1|$ lies between $2/q_{n+1}$ and $2\pi/q_{n+1}$, therefore the rate of growth of the denominators q_n describes how well θ may be approximated with rationals.

We recall a theorem due to Brjuno (1972):

Theorem 2.7 ([Bru]). *Let R be an analytic map with a periodic point $z_0 \in \hat{\mathbb{C}}$. Suppose that the multiplier of z_0 is $\lambda = e^{2\pi i\theta}$, and*

$$B(\theta) = \sum_n \frac{\log(q_{n+1})}{q_n} < \infty. \tag{2.4}$$

Then z_0 is a Siegel point.

Note that a quadratic polynomial with a fixed Sigel disk with rotation angle θ after an affine change of coordinates can be written as

$$P_\theta(z) = z^2 + e^{2\pi i\theta} z. \tag{2.5}$$

In 1987 Yoccoz [Yoc] proved the following converse to Brjuno's Theorem:

Theorem 2.8 ([Yoc]). *Suppose that for $\theta \in [0, 1)$ the polynomial P_θ has a Siegel point at the origin. Then $B(\theta) < \infty$.*

The numbers satisfying (2.4) are called Brjuno numbers; the set of all Brjuno numbers will be denoted $\mathcal{B}$. It is a full measure set which contains all Diophantine rotation numbers. In particular, the rotation numbers $[r_0, r_1, \ldots]$ of *bounded type*, that is with $\sup r_i < \infty$ are in $\mathcal{B}$. The sum of the series (2.4) is called the Brjuno function. For us a different characterization of $\mathcal{B}$ will be more useful. Inductively define $\theta_1 = \theta$ and $\theta_{n+1} = \{1/\theta_n\}$. In this way,

$$\theta_n = [r_{n-1}, r_n, r_{n+1}, \ldots].$$

We define the *Yoccoz's Brjuno function* as

$$\Phi(\theta) = \sum_{n=1}^{\infty} \theta_1 \theta_2 \cdots \theta_{n-1} \log \frac{1}{\theta_n}.$$

One can verify that

$$B(\theta) < \infty \Leftrightarrow \Phi(\theta) < \infty.$$

The value of the function Φ is related to the size of the Siegel disk in the following way.

Definition 2.5. Let (U, u) be a simply-connected subdomain of $\mathbb{C}$ with a marked interior point. Consider the unique conformal isomorphism $\phi : \mathbb{D} \mapsto U$ with $\phi(0) = u$, and $\phi'(0) > 0$. The *conformal radius of* (U, u) is the value of the derivative $r(U, u) = \phi'(0)$.

Let $P(\theta)$ be a quadratic polynomial with a Siegel disk $\Delta_\theta \ni 0$. The *conformal radius of the Siegel disk* Δ_θ is $r(\theta) = r(\Delta_\theta, 0)$. For all other $\theta \in [0, \infty)$ we set $r(\theta) = 0$, and $\Delta_\theta = \{0\}$.

By the Koebe 1/4 Theorem of classical complex analysis, the radius of the largest Euclidean disk around u which can be inscribed in U is at least $r(U, u)/4$.
We note that one has the following (see e.g. [BY]):

Proposition 2.9. *The conformal radius of a quadratic Siegel disk varies continuously with respect to the Hausdorff distance on Julia sets.*

Yoccoz [Yoc] has shown that the sum

$$\Phi(\theta) + \log r(\theta)$$

is bounded below independently of $\theta \in \mathcal{B}$. Recently, Buff and Chéritat have greatly improved this result by showing that:

Theorem 2.10 ([BC]). *The function $\theta \mapsto \Phi(\theta) + \log r(\theta)$ extends to $\mathbb{R}$ as a 1-periodic continuous function.*

In [BBY] we obtain the following result on computability of quadratic Siegel disks:

Theorem 2.11. *The following statements are equivalent:*

(I) *the Julia set $J(P_\theta)$ is computable;*
(II) *the conformal radius $r(\theta)$ is computable;*
(III) *the inner radius $\inf_{z\in\partial\Delta_\theta}|z|$ is computable.*

We note that when θ is not a Brjuno number, the quantities in (II) and (III) are each equal to zero, and the claim is simply that $J(P_\theta)$ is computable in this case.

We will make use of the following Lemma which bounds the variation of the conformal radius under a perturbation of the domain. It is a direct consequence of the Koebe Theorem (see e.g. [RZ] for a proof).

Lemma 2.12. *Let U be a simply-connected subdomain of $\mathbb{C}$ containing the point 0 in the interior. Let $V \subset U$ be a subdomain of U. Assume that $\partial V \subset B_\epsilon(\partial U)$. Then*

$$0 < r(U,0) - r(V,0) \le 4\sqrt{r(U,0)}\sqrt{\epsilon}.$$

2.4. **Outline of the construction.** We can now outline the idea of our construction. The discussion below is rather sketchy and suffers from obvious logical deficiencies, however, it presents the construction in a simple to understand form. Consider the oracle Turing machines M^ϕ with ϕ representing the parameter θ in P_θ. Since there are only countably many Turing machines, we may order these machines in a sequence M_1^ϕ, $M_2^\phi, \ldots$ We denote S_i the domain on which M_i^ϕ computes $J(P_\theta)$ properly. We thus have that for each i, the function $J : \theta \mapsto J(P_\theta)$ is continuous on S_i.
Let us start with a machine $M_{n_1}^\phi$ which computes $J(P_{\theta_*})$ for $\theta_* = [1,1,1,\ldots]$. If any of the digits r_i in this infinite continued fraction is changed to a sufficiently large $N \in \mathbb{N}$, the conformal radius of the Siegel disk will become small. For $N \to \infty$ the Siegel disk will implode and its center will become a parabolic fixed point in the Julia set (see [Do2]).

If we are careful, we may select $i_1 > 1$ and $N_1 >> 1$ in such a way, that for θ_1 given by the continued fraction where all digits are ones except $r_{i_1} = N_1$ we have

$$r(\theta_*)(1-1/4) < r(\theta_1) < r(\theta_*)(1-1/8). \tag{2.6}$$

By the Koebe 1/4-Theorem, there exists $\ell_1 > 0$ such that the distance between the two Julia sets

$$d_H(J(P_{\theta_*}), J(P_{\theta_1})) > 2^{-\ell_1}.$$

To ensure that the machine $M^{\phi}_{n_1}$ will not be able to produce an accurate $2^{-\ell_1}$-approximation of $J(P_{\theta_1})$ faster than in the time $h(\ell_1)$ we simply select $i_1 > h(\ell_1)$. This guarantees that the TM will have to read at least $h(\ell_1)$ digits of the oracle ϕ to distinguish the two Julia sets, which takes the time $h(\ell_1)$.

To "fool" the machine $M^{\phi}_{n_2}$ we then change a digit r_{i_2} for $i_2 > i_1$ sufficiently far in the continued fraction of θ_1 to a large N_2. In this way, we will obtain a Brjuno number θ_2 for which

$$r(\theta_*)(1 - 1/4 - 1/8) < r(\theta_2) < r(\theta_*)(1 - 1/4). \tag{2.7}$$

Again, there exists ℓ_2 such that for any such Brjuno number, we have

$$d_H(J(P_{\theta_1}), J(P_{\theta_2})) > 2^{-\ell_2},$$

and we choose $i_2 > h(\ell_2)$. Continuing inductively, we arrive at the desired limiting Brjuno number θ_∞.

To convince the reader that this construction is not artificial, and not due to the peculiarities of the selected computation model let us recast it somewhat informally as follows. It is possible by an arbitrarily small perturbation of the parameter θ to cause a detectable disturbance in the picture of $J(P_\theta)$. To distinguish the picture of the new Julia set from the old one, in practice one needs to draw it with *arbitrary precision arithmetic*. That is, not only the input of the parameter (reading the oracle) will take a long time due to the number of significant digits, but also all the arithmetic manipulations with this parameter. Of course, the former consideration is already sufficient to prove the theorem.

3. Computing Noble Siegel Disks

The primary goal of the present paper is to show that there are computationally hard yet *computable* Julia sets with Siegel disks. To establish this computability we need a computability result for *noble* Siegel disks. The term "noble" is applied in the literature to rotation numbers of the form $[a_0, a_1, \ldots, a_k, 1, 1, 1, \ldots]$. The noblest of all is the golden mean $\gamma_* = [1, 1, 1, \ldots]$.

We show:

Lemma 3.1. *There is a Turing Machine M, which given a finite sequence of numbers $[a_0, a_1, \ldots, a_k]$ computes the conformal radius r_γ for the noble number $\gamma = [a_0, \ldots, a_k, 1, \ldots]$.*

We recall first how in [BBY] we have established computability of $J(P_{\gamma_*})$. The idea is to approximate the boundary of Δ_{γ_*} by the iterates of the critical point $c_{\gamma_*} = -e^{2\pi i \gamma_*}/2$. It is known that in this case the critical point itself is contained in the boundary. The renormalization theory for golden-mean Siegel disks (see [McM]) implies that the boundary Δ_{γ_*} is self-similar up to an exponentially small error. In particular, there exist explicit constants $C > 0$ and $\lambda > 1$ such that

$$d_H(\{P^i_{\gamma_*}(c_{\gamma_*}), i = 0, \ldots, q_n\}, \partial\Delta_{\gamma_*}) < C\lambda^{-n} \tag{3.1}$$

The same theory applies to noble Siegel disks, with the necessary correction that (3.1) (with the same values of C and λ) only holds for sufficiently large values of n. These are the values for which the effect of the (possibly large) values of a_i is not felt anymore, and the renormalizations of Δ_γ look like the renormalizations of Δ_{γ_*}.

The point is then in constructively estimating such a value n. While we do not wish to go into the details of renormalization theory, which is a subject of its own, rather than simply giving a general reference, we will sketch such an algorithm.

Noble (or more generally, bounded type) Sigel quadratic Julia sets may be constructed by means of quasiconformal surgery on a Blaschke product

$$f_\gamma(z) = e^{2\pi i \tau(\gamma)} z^2 \frac{z-3}{1-3z}.$$

This map homeomorphically maps the unit circle $\mathbb{T}$ onto itself with a single (cubic) critical point at 1. The angle $\tau(\gamma)$ is selected in such a way that the rotation number of the restriction $\rho(f_\gamma|_{\mathbb{T}}) = \gamma$. The surgery construction, which is due to Douady, Ghys, Herman, and Shishikura [Do1], replaces the map f_γ inside the unit disk by the rigid rotation $R_\gamma : z \mapsto e^{2\pi i \gamma} z$ (for an expository account see [YZ]). It goes as follows.

First, the *universal real a priori bounds* for the renormalization theory (see [YZ] again for an exposition) give explicit upper and lower estimates on the size of the gaps between the points

$$\Omega_n = \{f_\gamma^i(1),\ i = 0, \dots q_n\}$$

on the unit circle. These gaps eventually decay exponentially in n with a universal bound, however, they can vary quite widely, depending on the size of a_i's.

Next step is to identify the orbit $\{f_\gamma^i(1)\}$ with the orbit $\{R_\gamma^i(1)\}$. The identification extends to a quasisymmetric map of the circle, whose Douady-Earle extension to $\mathbb{D}$ we denote H.

The above bounds yield a constructive estimate on the quasiconformal dilatation of the Douady-Earle map. The map H in turn induces a Beltrami form on $\mathbb{D}$, which we then propagate by $(f_\gamma)^*$ to an invariant Beltrami differential on $\hat{\mathbb{C}}$. Integrating it by the Measurable Riemann Mapping Theorem, we obtain a normalized quasiconformal map Ψ. Note, that we have arrived at this point with a constructive estimate on the distortion of Ψ.

Finally, it remains to note that $\Psi \circ f_\gamma \circ \Psi^{-1} = P_\gamma$. Given that

$$\Psi(\Omega_n) = \{P_\gamma^i(c_\gamma), i = 0, \dots, q_n\}, \text{ and } \Psi(\mathbb{T}) = \Delta_\gamma,$$

we have the desired estimate (3.1) for sufficiently large n and a computable bound on n.

To complete the algorithm, use any constructive algorithm for producing the Riemann mapping of a planar region, and the estimate of Lemma 2.12 to compute the desired conformal radius.

4. Making Small Changes to Φ and r

The next two lemmas are proved in [BY]. We only reproduce their statements.

For a number $\gamma = [a_1, a_2, \ldots] \in \mathbb{R} \setminus \mathbb{Q}$ we denote

$$\alpha_i(\gamma) = \cfrac{1}{a_i + \cfrac{1}{a_{i+1} + \cfrac{1}{a_{i+2} + \ldots}}},$$

so that

$$\Phi(\gamma) = \sum_{n\geq 1} \alpha_1(\gamma)\alpha_2(\gamma)\ldots\alpha_{n-1}(\gamma)\log\frac{1}{\alpha_n(\gamma)}.$$

We then have

Lemma 4.1. *For any initial segment $I = (a_0, a_1, \ldots, a_n)$, write $\omega = [a_0, a_1, \ldots, a_n, 1, 1, 1, \ldots]$. Then for any $\varepsilon > 0$, there is an $m > 0$ and an integer N such that if we write $\beta = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots]$, where the N is located in the $n+m$-th position, then*

$$\Phi(\omega) + \varepsilon < \Phi(\beta) < \Phi(\omega) + 2\varepsilon.$$

Lemma 4.2. *For ω as above, for any $\varepsilon > 0$ there is an $m_0 > 0$, which can be computed from $(a_0, a_1, \ldots, a_n)$ and ε, such that for any $m \geq m_0$, and for any tail $I = [a_{n+m}, a_{n+m+1}, \ldots]$ if we denote*

$$\beta^I = [a_1, a_2, \ldots, a_n, 1, 1, \ldots, 1, a_{n+m}, a_{n+m+1}, \ldots],$$

then

$$\sum_{i\geq n+m} \alpha_1(\omega)\alpha_2(\omega)\ldots\alpha_{i-1}(\omega)\log\frac{1}{\alpha_i(\omega)} < \varepsilon,$$

and

$$\sum_{i=1}^{n+m-1} \left| \alpha_1(\beta^I)\ldots\alpha_{i-1}(\beta^I)\log\frac{1}{\alpha_i(\beta^I)} - \alpha_1(\omega)\ldots\alpha_{i-1}(\omega)\log\frac{1}{\alpha_i(\omega)} \right| < \varepsilon.$$

We will need a *computable* version of Lemma 4.1 for modifying the conformal radius of the corresponding Julia set.

Lemma 4.3. *For any given initial segment $I = (a_0, a_1, \ldots, a_n)$ and $m_0 > 0$, write $\omega = [a_0, a_1, \ldots, a_n, 1, 1, 1, \ldots]$. Then for any $\varepsilon > 0$, we can uniformly compute $m > m_0$ and an integer N such that if we write $\beta = [a_0, a_1, \ldots, a_n, 1, 1, \ldots, 1, N, 1, 1, \ldots]$, where the N is located in the $n+m$-th position, we have*

$$r(\omega) - 2\varepsilon < r(\beta) < r(\omega) - \varepsilon, \tag{4.1}$$

and

$$\Phi(\beta) > \Phi(\omega). \tag{4.2}$$

Proof. We first show that such m and N *exist*, and then give and algorithm to compute them. By Lemma 4.1 we can increase $\Phi(\omega)$ by any controlled amound by modifying one term arbitrarily far in the expansion.

By Theorem 2.10, $f : \theta \mapsto \Phi(\theta) + \log r(\theta)$ extends to a continuous function. Hence for any ε_0 there is a δ such that $|f(x) - f(y)| < \varepsilon_0$ whenever $|x - y| < \delta$. In particular, there is an m_1 such that $|f(\beta) - f(\omega)| < \varepsilon_0$ whenever $m \geq m_1$.

This means that if we choose m large enough, a controlled increase of Φ closely corresponds to a controlled drop of r by a corresponding amount, hence there are $m > m_0$ and N such that (4.1) holds. (4.2) is satisfied almost automatically. The only problem is to *computably* find such m and N.

To this end, we apply Lemma 3.1. It implies that for any specific m and N we can compute $r(\beta)$. This means that we can find the suitable m and N, by enumerating all the pairs (m, N) and exhaustively checking (4.1) and (4.2) for all of them. We know that eventually we will find a pair for which (4.1) and (4.2) hold. □

5. Proving the Main Theorem

There are countably many oracle Turing Machines. Let us enumerate them in some arbitrary computable fashion $M_1^\phi, M_2^\phi, \ldots$ so that every machine appears infinitely many times in the enumeration. Recall that $r(\theta)$ is the conformal radius of the Siegel disk associated with the polynomial $P_\theta(z) = z^2 + e^{2\pi i\theta}z$, or zero, if θ is not a Brjuno number.

We will argue by induction. On each iteration i of the argument we shall maintain an initial segment $I_i = [a_0, a_1, \ldots, a_{N_i}]$ an interval $H_i = [l_i, r_i]$, and $\ell_i = \ell(H_i) = r_i - l_i$ such that the following properties are maintained:

$$r_i = r(\gamma_i), \text{ where } \gamma_i = [I_i, 1, 1, \ldots], \tag{5.1}$$

and

$$\text{For any } \beta = [I_i, t_{N_i+1}, t_{N_i+2}, \ldots] \text{ with } r(\beta) \in [l_i, r_i], \tag{5.2}$$

the machine M_i^ϕ requires at least the time $h(2\lceil -\log \ell_i \rceil + 1)$ to compute the $\frac{\ell_i^2}{2}$-approximation to $J(P_\beta)$.
Moreover, the intervals we construct are nested: $[l_i, r_i] \subset [l_{i-1}, r_{i-1}]$, and the sequence I_i contains I_{i-1} as the initial segment. The numbers $2\lceil -\log \ell_i \rceil$ form a strictly increasing sequence. We take $r_0 < 2$.
We also keep track of the *variation*
(5.3)
$$T_j = \sum_{i \geq 0} \left| \alpha_1(\gamma_{i+1})\alpha_2(\gamma_{i+1}) \ldots \alpha_{j-1}(\gamma_{i+1}) \log \frac{1}{\alpha_j(\gamma_{i+1})} - \alpha_1(\gamma_i)\alpha_2(\gamma_i) \ldots \alpha_{j-1}(\gamma_i) \log \frac{1}{\alpha_j(\gamma_i)} \right|$$

at each step of the process, in order to be able to pass to the limiting sequence $[a_0, a_1, a_2, a_3, \ldots]$ in the end.

For the basis of induction, set $I_0 = [1]$, $r_0 = r(\gamma_0)$ and $l_0 = r_0/2$, where $\gamma_0 = [1, 1, 1, \ldots]$. Then for $i = 0$ condition (5.1) holds by definition and condition (5.2) holds because it is empty.

The induction step. We now have the conditions (5.1) and (5.2) for some i and would like to extend them to $i+1$.

Consider the machine M^{ϕ}_{i+1}. Set $\ell_{i+1} = \frac{\ell_i}{20}$. Simulate M^{ϕ}_{i+1} on γ_i for at most $h(2\lceil -\log \ell_{i+1}\rceil + 1)$ steps to compute J_{γ_i} with precision $\frac{\ell^2_{i+1}}{2}$. The machine reads at most $h(2\lceil -\log \ell_{i+1}\rceil + 1)$ bits of the input, and we can compute m_0 such that this run does not distinguish between γ_i and $\gamma = [I_i, 1, 1, \ldots, 1, N_{m_0+1}, N_{m_0+2}, \ldots]$. There are two cases:
Case 1: M^{ϕ}_{i+1} does not terminate in the assigned time, or does not output a proper set. In this case, we proceed by setting $I_{i+1} = [I_i, 1, \ldots, 1]$ (with 1's up to position m_0), $\gamma_{i+1} = \gamma_i$, $r_{i+1} = r_i$, and $l_{i+1} = r_{i+1} - \ell_{i+1}$.

Case 2: M^{ϕ}_{i+1} outputs a set S. Compute the conformal radius $r(S)$. Elementary considerations of Koebe's theorem imply that for any quadratic Siegel disk, $r(\Delta) < 2$. Using the above consideration to bound the constant in Lemma 2.12, we know that for any Julia set $J(P_\omega)$ which is ℓ^2_{i+1}-accurately described by S, we have $|r(\omega) - r(S)| < 4\sqrt{2}\ell_{i+1} < 6\ell_{i+1}$. Again, there are two cases (if both hold, it doesn't matter which way to proceed):
Subcase 2a: $r_i - \ell_{i+1} > r(S) + 8\ell_{i+1}$. In this case we proceed by setting $I_{i+1} = [I_i, 1, \ldots, 1]$ (with 1's up to position m_0), $\gamma_{i+1} = \gamma_i$, $r_{i+1} = r_i$, and $l_{i+1} = r_{i+1} - \ell_{i+1}$.
Subcase 2b: $l_i + 2\ell_{i+1} < r(S) - 8\ell_{i+1}$. By Lemma 4.3, we can select $\gamma_{i+1} = [I_{i+1}, 1, 1, \ldots]$ by modifying γ_i at an arbitrarily far position, and set $r_{i+1} = r(\gamma_{i+1})$ so that $r_{i+1} \geq l_i + \ell_{i+1}$ and $[r_{i+1} - \ell_{i+1}, r_{i+1}] \cap [r(S) - 8\ell_{i+1}, r(S) + 8\ell_{i+1}] = \emptyset$. The number r_{i+1} is computable since it is the conformal radius of a noble Siegel disk. Set $l_{i+1} = r_{i+1} - \ell_{i+1}$. We see that the induction is maintained for these parameters.

A bound on the variation T_j is necessary to conclude:

Lemma 5.1 (Lemma 4.2, [BY]). *The sequence $\{\Phi(\gamma_i)\}$ converges to a limit*

$$\Phi(\lim \gamma_i) = \lim \Phi(\gamma_i).$$

The proof is based on an application of Fubini's Theorem. The parameter $\gamma = \lim \gamma_i$ is a computable Brjuno number, with

$$r(\gamma) = \lim r(\gamma_i),$$

by [BC]. The conformal radius $r(\gamma)$ is computable, since the convergence $r(\gamma_i) \to r(\gamma)$ is uniform. Thus J_{P_γ} is also computable by Theorem 2.11. By construction, it satisfies all of the required properties.

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