ON THE COMPUTATIONAL COMPLEXITY OF THE RIEMANN MAPPING

I. BINDER, M. BRAVERMAN, M. YAMPOLSKY

Abstract. In this paper we consider the computational complexity of uniformizing a domain with a given computable boundary. We give nontrivial upper and lower bounds in two settings: when the approximation of boundary is given either as a list of pixels, or by a Turing Machine.

1. Introduction

1.1. **Foreword.** Computational conformal mapping is prominently featured in problems of applied analysis and mathematical physics, as well as in engineering disciplines, such as image processing. In this paper we address the theoretical foundations of numerically approximating the conformal mapping between two planar domains. We obtain a lower bound on the computational complexity of an algorithm solving this problem, and show that this bound is almost sharp. To achieve the latter, we present a very space-efficient probabilistic algorithm for constructing such a mapping.

Acknowledgments. The second author would like to thank Ker-I Ko for bringing the problem to his attention during CCA'04.

The authors are grateful to Stephen Cook for his numerous helpful suggestions during the preparation of the paper.

1.2. **Background in computational complexity theory.** We present here some basic definitions and results from the computational complexity theory. A more comprehensive discussion can be found in [Sip, Papa].

The primary goal of the computational complexity theory is to classify different computational problems into *complexity classes* according to their computational hardness. The basic abstract object here is a *Turing Machine* which for most purposes can be thought of as a program in any programming language.

The complexity class **P** includes problems that are computable in time polynomial in the length of the input. Those are thought of as the "relatively easy" problems. Examples of problems in **P** include arithmetic operations, finding a shortest path in a graph and primality testing. "Difficult" problems, such as factoring integers or computing the optimal strategy for playing "Go" on an $n \times n$ board, are generally thought not to be in **P**. Whether a problem is in **P** or not is usually a good criterion in assessing the true hardness of it. By an analogy with **P** one can define the class **EXP** of problems solvable in time 2^{n^c} for some c on input length

Date: January 31, 2007.

The first and third authors were partially supported by NSERC Discovery grants. The second author was partially supported by NSERC Post-graduate Scholarship.

n. Playing "Go" optimally is in **EXP**, since we can easily enumerate all possible games and compute an optimal path in time $2^{O(n^2)}$. Using a diagonalization argument, it is not hard to see that $\mathbf{P} \subsetneq \mathbf{EXP}$ (see e.g. [Sip]).

The complexity class **NP** contains problems that are easy to verify, but may be hard to guess. More precisely, a predicate $Q(x)$ is in **NP**, if there is a poly-time computable predicate $R(x, y)$, where y has length polynomial in the length of x, such that $Q(x) = \exists y\ R(x, y)$. By an exhaustive search for y one sees that $\mathbf{NP} \subset \mathbf{EXP}$. There is a subclass of **NP** called the **NP**-complete problems, or **NPC**. Problems in **NPC** have the property of being the "hardest" in **NP**: if one could solve *any* problem in **NPC** in polynomial time, then one could solve *all* **NP** problems in polynomial time.

One of the most famous **NP**-complete problems is the *satisfiability* problem SAT. The problem is the following: given a propositional formula $\phi(y)$ does it have a truth assignment y_T such that $\phi(y_T) = 1$. An example of a problem in **NP** that is thought to be hard but not **NP**-complete is the following. Given a pair of numbers $m < n$, determine whether n has a divisor between 2 and m. This problem can be used to factor integers. It is in **NP** since it can be formulated as $\exists k\ (1 < k < m) \wedge k \mid n$. It is one of the Clay $1,000,000 questions of whether $\mathbf{P} = \mathbf{NP}$.

A bigger class of problems is the class #**P**. It is the class of problems which are equivalent to counting the number of satisfying assignments for a given propositional formula – this natural complete problem for this class is denoted by $\#SAT$. Obviously $\mathbf{NP} \subset \#\mathbf{P}$, since to solve SAT we only need to know whether the number of its satisfying assignments is bigger than 0 or not, which is easier than actually determining this number.

The next class of problems is the class **PSPACE** – the class of problems solvable in *space* polynomial in the input size. It is easy to see that all the classes mentioned above are in **PSPACE**. On the other hand, $\mathbf{PSPACE} \subset \mathbf{EXP}$, since a machine with $p(n)$ memory bits can have at most $2^{p(n)}$ different configurations, and can run for at most $2^{p(n)}$ steps without getting into an infinite loop.

The class of problems solvable in *logarithmic* space is a class of problems that are solvable in space $O(\log n)$ for input size n. Here the input and the output are read-only and write-only respectively. This class is denoted by L. By the same reasoning as $\mathbf{PSPACE} \subset \mathbf{EXP}$, we have $\mathbf{L} \subset \mathbf{P}$. A *randomized* version of **L** are the problems that can be solved correctly with error probability $< 1/n$ in space $O(\log n)$ and time poly(n). This class is called **BPL**.

Overall, we have the following chain of inclusions:

$$\mathbf{L} \subseteq \mathbf{BPL} \subseteq \mathbf{P} \subseteq \mathbf{NP} \subseteq \#\mathbf{P} \subseteq \mathbf{PSPACE} \subseteq \mathbf{EXP}.$$

By diagonalization, $\mathbf{BPL} \neq \mathbf{PSPACE}$, and $\mathbf{P} \neq \mathbf{EXP}$. No other separations are known.

In recent years, some progress has been made in *derandomizing* **BPL**. In other words, showing that there is a deterministic algorithm that requires not much more computational resources. We will need the following recent result on derandomization:

Theorem 1.1. [Nis] *There exists a deterministic algorithm for the following problem:*

Input: *An $n \times n$ transition probability matrix M, an integer t, and a rational ϵ.*

Output: *A matrix A such that $||A - M^t|| \leq \epsilon$.*

The algorithm runs in time poly(N) and space $O(\log^2 N)$, where $N = n^2 + t + \epsilon^{-1}$.

We will also need two *circuit complexity* classes. A circuit consist of inputs, logical gates, and an output. The gates are usually NOT (one input, one output), AND, and OR. The latter gates can either have two, or unboundedly many inputs. In the discussion below, any number of inputs is allowed. The *size* of a circuit is the number of gates used. The *depth* of a circuit is the number of gates on the longest path from an input to the output. It is known that any boolean function $f : \{0,1\}^n \to \{0,1\}$ can be computed by a circuit of size $O(2^n/n)$. It is not hard to see that functions in P are computable by polynomial size circuits. The class $\mathbf{AC^0}$ is the class of functions that are computable by a family of circuits (one for each input size), that have constant depth and polynomial size. This is one of the very few complexity classes for which non-diagonalization lower bounds exist. In particular it has been shown that computing the parity of the number of 1's in a string cannot be done in $\mathbf{AC^0}$ (see, for example, [FSS]). A more general problem that cannot be done in $\mathbf{AC^0}$ is the *majority problem* MAJ_n: given a string $x \in \{0,1\}^n$, $\mathrm{MAJ}_n(x)$ is 1 if and only if the majority of the entries in x are 1.

1.3. **Computational complexity of sets.** We review the definition and the basic properties of computable sets. We refer the reader to [BW, Wei, RW, Brav] for a more comprehensive exposition.

Intuitively, we say the time complexity of a set S is $t(n)$ if it takes time $t(n)$ to decide whether to draw a pixel of size 2^{-n} in the picture of S. Mathematically, the definition is as follows:

Definition 1.1. A set T is said to be a 2^{-n}-picture of a bounded set S if:

(i) $S \subset T$, and
(ii) $T \subset B(S, 2^{-n}) = \{x \in \mathbb{R}^2 \ : \ |x - s| < 2^{-n} \text{ for some } s \in S\}$.

Definition 1.1 means that T is a 2^{-n}-approximation of S with respect to the *Hausdorff metric*, given by

$$d_H(S,T) := \inf\{r : S \subset B(T,r) \text{ and } T \subset B(S,r)\}.$$

Suppose we are trying to generate a picture of a set S using a union of round pixels of radius 2^{-n} with centers at all the points of the form $\left(\frac{i}{2^n}, \frac{j}{2^n}\right)$, with i and j integers. In order to draw the picture, we have to decide for each pair (i,j) whether to draw the pixel centered at $\left(\frac{i}{2^n}, \frac{j}{2^n}\right)$ or not. We want to draw the pixel if it intersects S and to omit it if some neighborhood of the pixel does not intersect S. Formally, we want to compute a function

$$f_S(n, i/2^n, j/2^n) = \begin{cases} 1, & B((i/2^n, j/2^n), 2^{-n}) \cap S \neq \emptyset \\ 0, & B((i/2^n, j/2^n), 2 \cdot 2^{-n}) \cap S = \emptyset \\ 0 \text{ or } 1, & \text{in all other cases} \end{cases} \tag{1.1}$$

The time complexity of S is defined as follows.

Definition 1.2. A bounded set S is said to be computable in time $t(n)$ if there is a function $f(n, \bullet)$ satisfying (1.1) which runs in time $t(n)$. We say that S is poly-time computable if there is a polynomial p, such that S is computable in time $p(n)$.

Computability of sets in bounded space is defined in a similar manner. There, the amount of *memory* the machine is allowed to use is restricted.

To see why this is the "right" definition, suppose we are trying to draw a set S on a computer screen which has a 1000×1000 pixel resolution. A 2^{-n}-zoomed in picture of S has $O(2^{2n})$ pixels of size 2^{-n}, and thus would take time $O(t(n) \cdot 2^{2n})$ to compute. This quantity is exponential in n, even if $t(n)$ is bounded by a polynomial. But we are drawing S on a finite-resolution display, and we will only need to draw $1000 \cdot 1000 = 10^6$ pixels. Hence the running time would be $O(10^6 \cdot t(n)) = O(t(n))$. This running time is polynomial in n if and only if $t(n)$ is polynomial. Hence $t(n)$ reflects the 'true' cost of zooming in.

1.4. **Background in complex analysis.** To make the paper self-contained, we list here a few results from complex analysis that will be used later. We refer to [Ahl, Dur, Pom] for a more comprehensive discussion.

Let $\Omega \subsetneq \mathbb{C}$ be a simply-connected planar domain with $w \in \Omega$. *Riemann Uniformization Theorem* states that there is unique conformal map ψ of Ω onto the unit disk $\mathbb{D}$ with $\psi(w) = 0$, $\psi'(0) > 0$. The number $r(\Omega, w) = 1/\psi'(0)$ is called the *conformal radius* of Ω. Roughly speaking, $r(\Omega, w)$ measures the size of Ω as viewed from w:

Proposition 1.2 (Koebe's Theorem)**.** *In this notation we have*

$$\mathrm{dist}(w, \partial\Omega) \geq \frac{r(\Omega, w)}{4}.$$

We note the following basic monotonicity property of the conformal radius:

Lemma 1.3. *If $\Omega_1 \subset \Omega_2$, $w \in \Omega_1$, then $r(\Omega_1, w) \leq r(\Omega_2, w)$.*

By a theorem of Carathéodory (see e.g. [Pom]), if the boundary $\partial\Omega$ is a Jordan curve, then the map ψ can be extended to a homeomorphism between the closure of Ω and the closed unit disk $\mathrm{cl}(\mathbb{D})$.

Let $z^* = 1/\overline{z}$ be the inversion of z with respect to the unit circle $\{|z| = 1\}$. We will make use of the following particular case of the *Reflection Principle*:

Lemma 1.4. *If $J \subset \{|z| = 1\}$ is an open arc, and ϕ is a continuous map on $\mathbb{D} \cup J$ which is analytic on $\mathbb{D}$, and $\phi(J) \subset \{|z| = 1\}$, then the map Φ defined by*

$$\Phi(z) = \begin{cases} \phi(z), & |z| < 1 \text{ and } z \in J \\ \phi^*(z^*), & |z| > 1 \end{cases} \tag{1.2}$$

is analytic at the domain $\mathbb{D} \cup \{|z| > 1\} \cup J$.

In particular, if ϕ is a conformal map of $\mathbb{D}$ onto a domain $\Omega \subset \mathbb{D}$ with Jordan boundary, and K is an open arc, $K \subset \partial\Omega \cap \{|z| = 1\}$, then Φ is a conformal map of $\mathbb{D} \cup \{|z| > 1\} \cup J$, where $J = \phi^{-1}(K)$ (ϕ is extendable to $\mathrm{cl}(\mathbb{D})$ by Carathéodory theorem).

Let now Ω be a domain with the boundary $\partial\Omega$ consisting of finitely many Jordan curves. Let f be a continuous function on $\partial\Omega$. A function $u : \mathrm{cl}(\Omega) \to \mathbb{C}$ is a *solution for the Dirichlet problem with the boundary data f*, if

- u is continuous in $\mathrm{cl}\,\Omega$,
- u is harmonic in Ω ($\Delta u = \partial_{xx} u + \partial_{yy} u = 0$), and
- $u(z) = f(z)$ for $z \in \partial\Omega$.

For any f such a solution exists and is unique. Moreover, there exists a unique family of measures $\omega_{w,\Omega}$ on $\partial\Omega$ such that for any $f \in C(\partial\Omega)$,

$$u(w) = \int_{\partial\Omega} f(z)\, d\omega_{w,\Omega}(z).$$

The measure $\omega_{w,\Omega}$ is called a *harmonic measure*. If one fixes $K \subset \partial\Omega$, the function $w \mapsto \omega_{w,\Omega}(K)$ is harmonic in Ω.
If Ω is simply-connected, then for a set $K \subset \partial\Omega$, we have

$$\omega_{w,\Omega}(K) = \frac{1}{2\pi} \operatorname{length}(\psi(K)),$$

where ψ is the Riemann map of Ω onto $\mathbb{D}$ with $\psi(w) = 0$.

The Dirichlet problem can be solved probabilistically. Namely, for $w \in \Omega$, let $B_w(t)$ be the two-dimensional Brownian motion started at w, and the exit time be defined by

$$T = \inf\{t \;:\; B_w(t) \notin \Omega\}.$$

Then the solution of the Dirichlet problem is given by the following formula of Kakutani (see e.g. [GM]):

$$u(w) = \mathbf{E}[f(B_w(T))].$$

Note that the harmonic measure for a set $K \subset \partial\Omega$ is now given by

$$\omega_{w,\Omega}(K) = \mathbf{P}[B_w(T) \in K].$$

We will make use of the *Maximum Principle for harmonic functions* (see [Ahl]):

Lemma 1.5. *If $u_1(z)$ and $u_2(z)$ are two functions which are harmonic in Ω, continuous on the whole* $\mathrm{cl}(\Omega)$, *and $u_1(z) \geq u_2(z)$ for $z \in \partial\Omega$, then $u_1(z) \geq u_2(z)$ for all $z \in \Omega$.*

An easy consequence is the *Monotonicity Property of the harmonic measure* (see [Pom]):

Corollary 1.6. *If $w \in \Omega_1 \subset \Omega_2$, $K \subset \partial\Omega_1 \cap \partial\Omega_2$, then*

$$\omega_{w,\Omega_1}(K) \leq \omega_{w,\Omega_2}(K).$$

We will also make use of a Distortion Theorem for conformal maps(see [Dur]):

Theorem 1.7. *If ϕ is conformal in the disk $\{z \;:\; |z - w| < r\}$, then*

$$|\phi'(w)| \frac{r^2|z-w|}{(r+|z-w|)^2} \leq |\phi(z) - \phi(w)| \leq |\phi'(w)| \frac{r^2|z-w|}{(r-|z-w|)^2} \tag{1.3}$$

and

$$r^2|\phi'(w)| \frac{r-|z-w|}{(r+|z-w|)^3} \leq |\phi'(z)| \leq r^2|\phi'(w)| \frac{r+|z-w|}{(r-|z-w|)^3} \tag{1.4}$$

1.5. **Results.** In Section 2 we propose a new algorithm for computing the Riemann map. We use the random walks solution to the general Dirichlet problem to produce a solution to the uniformization problem. This gives an extremely space-efficient algorithm.

The formulation of the theorem will depend on how the boundary of the uniformized domain Ω is specified for our algorithm. Since the domain Ω we consider is computable, there exists a Turing machine $M(n)$ which for a given n computes a function (1.1). Our algorithm may then query $M(n)$ for different values of $(i/2^n, j/2^n)$ to ascertain whether this particular dyadic rational point lies within one-pixel distance from $\partial\Omega$. A formal way of saying this is that our algorithm will have an access to an *oracle* for a function given by (1.1).

Theorem 1.8. *There is an algorithm A that computes the uniformizing map in the following sense.*

Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. $\partial\Omega$ is provided to A by an oracle representing it in the sense of equation (1.1). *Then A computes the absolute values of the uniformizing map $\phi : (\Omega, w_0) \to (\mathbb{D}, 0)$ with precision 2^{-n} in space bounded by $C \cdot n^2$, and time $2^{O(n)}$, where C depends only on the diameter of Ω and $d(w_0, \partial\Omega)$. Furthermore, the algorithm computes the value of $\phi(w)$ with precision 2^{-n} as long as $|\phi(w)| < 1 - 2^{-n}$. Moreover, A queries $\partial\Omega$ with precision of at most $2^{-O(n)}$.*

In particular, if $\partial\Omega$ is polynomial space computable in space n^a for some constant $a \geq 1$ and time $T(n) < 2^{O(n^a)}$, then A can be used to compute the uniformizing map in space $C \cdot n^{\max(a,2)}$ and time $2^{O(n^a)}$.

In the scale where the entire boundary is given to us explicitly, and not by an oracle for it, we have the following.

Theorem 1.9. *There is an algorithm A' that computes the uniformizing map in the following sense.*

Let Ω be a bounded simply-connected domain, and $w_0 \in \Omega$. Suppose that for some $n = 2^k$, $\partial\Omega$ is given to A' with precision $\frac{1}{n}$ by $O(n^2)$ pixels. Then A' computes the absolute values of the uniformizing map $\phi : (\Omega, w_0) \to (\mathbb{D}, 0)$ within an error of $O(1/n)$ in randomized space bounded by $O(k)$ and time polynomial in $n = 2^k$ (that is, by a **BPL**(n)*-machine). Furthermore, the algorithm computes the value of $\phi(w)$ with precision $1/n$ as long as $|\phi(w)| < 1 - 1/n$.*

In Section 3, we show that even if the domain we are uniformizing is very simple computationally, the complexity of the uniformization can be quite high. Moreover, it might be difficult to compute the conformal radius of the domain.

More specifically, the following theorems are established in the Section 3.

Theorem 1.10. *Suppose there is an algorithm A that given a simply-connected domain Ω with a linear-time computable boundary and an inner radius $> \frac{1}{2}$ and a number n computes the first $20n$ digits of the conformal radius $r(\Omega, 0)$, then we can use one call to A to solve any instance of a $\#SAT(n)$ with a linear time overhead.*

In other words, #**P** *is poly-time reducible to computing the conformal radius of a set.*

Theorem 1.11. *Consider the problem of computing the conformal radius of a simply-connected domain Ω, where the boundary of Ω is given with precision $\frac{1}{n}$ by an explicit collection of $O(n^2)$ pixels.*

Denote the problem of computing the conformal radius with precision $\frac{1}{n^c}$ *by* $CONF(n, n^c)$ *Then* MAJ_n *is* $\mathbf{AC^0}$ *reducible to* $CONF(n, n^c)$ *for any* $0 < c < 1/2$.

1.6. **Comparison with known results.** The first constructive proof of the Riemann Uniformization Theorem is due to Koebe [Koebe], and dates to the early 1900's. Formal proofs of the constructive nature of the Theorem which follow Koebe's argument under various computability conditions on the boundary of the domain are numerous in the literature (see e.g. [Cheng, BB, Zhou, Hert]). In particular, Zhou [Zhou] and Hertling [Hert] give constructive proofs under computability conditions on the boundary similar to those used by us. The question of complexity bounds on the construction was raised, in particular, in most of the works quoted above. However, the only result known to us was announced by Chou in [Chou]. He states that in the case when the boundary is poly(n) computable, the problem of computation of the mapping is in **EXPSPACE**(n).

From the practical (that is, applied) point of view, the most computationally efficient algorithm used nowadays to calculate the conformal map is the "Zipper", invented by Marshall (see [Mar]). The effectiveness of this algorithm was recently studied by Marshall and Rohde in [MR]. The "Zipper", however, falls beyond the theoretical upper bound on the complexity of this problem, which we establish in Section 2: in the settings of the Theorem 1.8, it computes the uniformizing map in space $2^{O(n^a)}$ and time $2^{O(n^a)}$, and thus belongs to the complexity class **EXP**. It is reasonable to expect then, that an algorithm can be found in class **PSPACE** which is more practically efficient than "Zipper".

2. Computing the uniformization in polynomial space

Let Ω be a bounded simply-connected planar domain, $\operatorname{diam}(\Omega) = 1$ and let $K \subset \Omega$ be a fixed compact set with smooth boundary with $\operatorname{dist}(K, \partial\Omega) > 10 \cdot 2^{-n}$. First we discuss a probabilistic algorithm for solving the Dirichlet problem in the domain $\Omega \setminus K$ with precision 2^{-n}.

2.1. **General Dirichlet problem.** The discrete analogue of the Dirichlet problem can be defined as follows. For $H \subset h\mathbb{Z}^2$ ($h > 0$), the interior of H is defined by $\operatorname{Int}(H) = \{a \in H : a \pm h, a \pm ih \in H\}$. The boundary of H is defined by $\partial H = h\mathbb{Z}^2 \setminus \big(\operatorname{Int}(H) \cup \operatorname{Int}(h\mathbb{Z}^2 \setminus H)\big)$. We say that a function u defined on $H \subset h\mathbb{Z}^2$ is *discrete harmonic* if for any $a \in \operatorname{Int}(H)$ we have

$$u(a) = 1/4(u(a + h) + u(a - h) + u(a + ih) + u(a - ih)).$$

Let B_n^w be the standard Random Walk (cf [Spi]) on $h\mathbb{Z}^2$ started at $w \in H$, where H is closed ($\partial H \subset H$). Let the exit time N be defined as $N = \min\{n : B_n^w \notin H\} - 1$. Let f be a function on ∂H. It is almost obvious that the function

$$u(w) = \mathbb{E}(f(B_N^w))$$

is discrete harmonic on H. This function is called the *solution for the Dirichlet problem with the boundary data* f (compare to the continuous case discussion in subsection 1.4).

Let now Ω be a domain with boundary $\partial\Omega$, $f \in C(\partial\Omega)$. For $h > 0$ define $H_h = \Omega \cap h\mathbb{Z}^2$. For $w \in \partial H_h$, let $f_h(w) = f(z)$, where z is one of the points on $\partial\Omega$ closest to w. u_h, the solution to the corresponding discrete Dirichlet problem, is called h-discrete solution to the initial continuous Dirichlet problem.

We need the following easy case of the approximating property of the h-discrete solutions (see [Spi], [Laa]).

Lemma 2.1. *Let Ω be a domain, f is locally constant, and takes only 0 and 1 values, and u is the solution of the corresponding Dirichlet problem. Let u_h be the h-discrete solution. Then if $\operatorname{dist}(w, \partial\Omega) > h$, then $|u(w) - u_h(w)| \leq 2\sqrt{h}$.*

Since the exit probabilities of a random walk can be computed by a $\mathbf{BPL}(h^{-1})$ machine; if the values of f and the boundary ∂H_h are given by an oracle, then u can be computed in the randomized space $O(-\log h)$ and time $O(h^{-2})$. Thus Lemma 2.1 immediately implies the following statement about the solution of the general Dirichlet problem:

Lemma 2.2. *There is a randomized algorithm D that computes a solution of the Dirichlet problem in the following sense.*

Let Ω be a bounded planar domain and $K \subset \Omega$ be a fixed compact set with smooth boundary and $\operatorname{dist}(K, \partial\Omega) > 10\cdot 2^{-n}$. Suppose that f is the function which is equal to 0 on $\partial\Omega$ and 1 on K. Then D computes the solution of the corresponding Dirichlet problem with precision 2^{-n}, 2^{-n}-away from $\partial\Omega \cup K$ in space $O(n)$, and time $2^{O(n)}$. The computation is done probabilistically, and outputs the correct value within an error of 2^{-n} with probability $> \frac{1}{2}$.

In particular, if both K and $\partial\Omega$ are computable in space n^a for some constant $a \geq 1$ and time $T(n) < 2^{O(n^a)}$. Then we can compute the solution of the Dirichlet problem for any point, which is at least 2^{-n} away from $\partial\Omega$ and K in space $O(n^a)$, and time $2^{O(n)}T(n)$.

2.2. **Conformal radius.** Let $w_0 \in \Omega$, and let ψ be the conformal mapping of Ω onto the unit disk $\mathbb{D}$ with $\psi(w_0) = 0$ and $\psi'(w_0) > 0$. Assume that $\partial\Omega$ is given to us up to distance 2^{-n} in Hausdorff metric and that $d(w_0, \partial\Omega) \geq 1/2$. As the first application of Lemma 2.2 let us give an algorithm for calculating $|\psi'(w_0)|$ with precision 2^{-n} in space $O(n^a)$, and time $2^{O(n)}T(n)$. As before, Denote

$$w_1 = w_0 + e^{-n} \text{ and } K_1 = B(w_0, e^{-2n})$$

Lemma 2.3. *Let h_1 be the solution of the following Dirichlet problem:*

$$\begin{cases} h_1(w) = 1, & |w - w_0| = e^{-2n} \\ h_1(w) = 0, & w \in \partial\Omega \\ \Delta h_1(w) = 0, & w \in \Omega \setminus K_1 \end{cases}$$

Then

$$\left| \log|\psi'(w_0)| - n\left(\frac{1 - 2h_1(w_1)}{1 - h_1(w_1)}\right) \right| \leq Cn\ e^{-n}$$

for some absolute constant C.

Proof. By the first statement of Theorem 1.7,

$$B(0, e^{-2n}/(1 + e^{-2n})^2 \psi'(w_0)) \subset \psi(K_1) \subset B(0, e^{-2n}/(1 + e^{-2n})^2 \psi'(w_0)) \tag{2.1}$$

Let $B_1 = \psi^{-1}\left(B(0, e^{-2n}(1 - 3e^{-2n})\psi'(w_0))\right)$ and $B_2 = \psi^{-1}\left(B(0, e^{-2n}(1 - 3e^{-2n})\psi'(w_0))\right)$. Since $1/(1 + e^{-2n})^2 < (1 - 3e^{-2n})$ and $(1 + 3e^{-2n}) < 1/(1 - e^{-2n})^2$, (2.1) implies

$$B_1 \subset K_1 \subset B_2 \tag{2.2}$$

The functions

$$H_1(w) = \frac{\log|\psi(w)|}{-2n + \log(1-3e^{-2n}) + \log(\psi'(w_0))}$$

and

$$H_2(w) = \frac{\log|\psi(w)|}{-2n + \log(1+3e^{-2n}) + \log(\psi'(w_0))}$$

are harmonic in $\Omega \setminus B_1$ and $\Omega \setminus B_2$ correspondingly, equal to 0 on $\partial\Omega$, and equal to 1 on the boundaries of B_1 and B_2 respectively. By the Maximum Principle, $H_1 \le h_1 \le H_2$, or, more explicitly,

$$\frac{\log|\psi(w)|}{-2n + \log(1-3e^{-2n}) + \log(\psi'(w_0))} \le h_1(w) \le \frac{\log|\psi(w)|}{-2n + \log(1+3e^{-2n}) + \log(\psi'(w_0))} \tag{2.3}$$

Another application of the same distortion theorem gives

$$e^{-n}(1-3e^{-n})\psi'(w_0) \le |\psi(w_1)| \le e^{-n}(1+3e^{-n})\psi'(w_0). \tag{2.4}$$

Evaluating both sides of the inequality 2.3 at the point w_1 using 2.4 gives the statement of the lemma. □

It now follows from Lemma 2.2 that we can compute $|\psi'(0)|$ with the same complexity constraints as in Lemma 2.2.

2.3. **The Riemann map.** Let h_1, K_1 be as in the previous section.

Lemma 2.4. *Let* $e^{-n} < |w - w_0|$. *Then*

$$\big|\log|\psi(w)| - h_1(w)(\log(\psi'(w_0)) - 2n)\big| \le 3 \cdot e^{-n}.$$

Proof. By the equation (2.3),

$$h_1(w)\log(1-3e^{-n}) \le \log|\psi(w)| - h_1(w)(\log(\psi'(w_0)) - 2n) \le h_1(w)\log(1+3e^{-n})$$

To prove the lemma it is enough to notice that $h_1(w) \le 1$ and $|\log(1+x)| \le |x|$. □

Using Lemmas 2.3 and 2.2, we see that $|\psi(w)|$ is computable with the same restrictions as in Lemma 2.2, provided that $\text{dist}(w, \partial\Omega) > e^{-n}$ and $|w - w_0| > e^{-n}$.

Now we have to compute $\arg\big(\psi(w)\big)$. To achieve this, we introduce another Dirichlet problem. Let $K_2 = B(w_0 + e^{-2n}, e^{-4n})$, and let h_2 be the solution of the following Dirichlet problem:

$$\begin{cases} h_2(w) = 1, & |w - w_0 - e^{-2n}| = e^{-4n} \\ h_2(w) = 0, & w \in \partial\Omega \\ \Delta h_2(w) = 0, & w \in \Omega \setminus K_2 \end{cases}$$

Let

$$\tilde{\psi}(w) = \frac{\psi(w) - e^{-2n}\psi'(0)}{1 - \psi(w)e^{-2n}\psi'(0)}$$

be another conformal map from Ω onto $\mathbb{D}$, with $w_2 = \tilde{\psi}^{-1}(0)$. By Distortion Theorem 1.7,

$$|w_2 - w_0 - e^{-2n}| \le C \cdot e^{-4n}$$

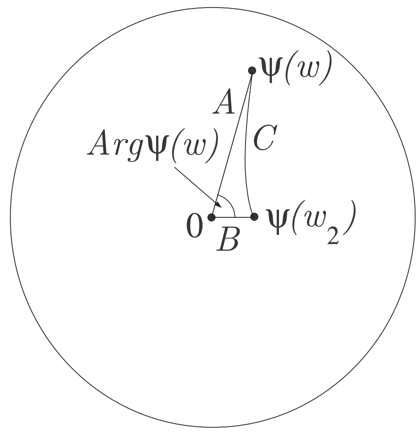

FIGURE 1. Computing $\arg\big(\psi(w)\big)$

for some absolute constant C. As in Lemma 2.4,

$$\left|\log|\tilde{\psi}(w)| - h_2(w)\log\psi'(w_0) - 4n\right| \leq C \cdot e^{-2n}.$$

Now we use a standard formula from hyperbolic trigonometry (see [Thu])

$$\cos\arg\psi(w) = \frac{\cosh C - \cosh A \cosh B}{\sinh A \sinh B},$$

where

$$A = \log\frac{1+|\psi(w)|}{1-|\psi(w)|},\ B = \log\frac{1+e^{-2n}\psi'(0)}{1-e^{-2n}\psi'(0)}, \text{ and } C = \log\frac{1+|\tilde{\psi}(w)|}{1-|\tilde{\psi}(w)|}.$$

See Figure 1.

Note that $\sinh B \sim e^{-2n}$, $\sinh A > e^{-n}$ when $|w| > e^{-n}$, $\cosh B - 1 \sim e^{-2n}$. Using the error estimate in the Lemma 2.4, we obtain that the formula allows us to compute $\cos\arg\psi(w)$ up to e^{-n}, provided that $|\psi(w)| < 1 - e^{-2n}$.

Using the same argument for computing $\cos\arg(\psi(w)/i)$, we can completely determine the value of $\arg\phi(w)$.

Now we can give an algorithm which satisfies the conditions of Theorems 1.8 and 1.9.

Proof of Theorems 1.8 and 1.9. We can create a $\text{poly}(n) \times \text{poly}(n)$ matrix M representing the transition probabilities between the $\text{poly}(n)$ possible states of the random walk. Simulating the random walk for $t = \text{poly}(n)$ steps amounts to approximating M^t. The required precision is also inverse polynomial in n. By Theorem 1.1, this can be done in time polynomial in n, and space $O(\log^2 n)$, which imply Theorem 1.9. By changing the scale, and replacing n with 2^n, we obtain Theorem 1.8. □

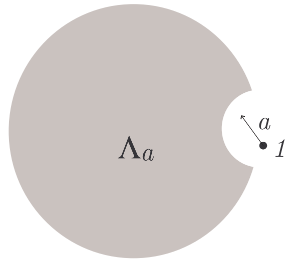

FIGURE 2. Λ_a

3. Lower bounds on the complexity of uniformization

In this section we establish Theorems 1.10 and 1.11.

Let us first remark that, by Distortion Theorem 1.7, any algorithm computing values of the uniformization map will also compute the conformal radius with the same precision.

Let Λ_a be the domain $\mathbb{D}\setminus\{|z-1|\leq a\}$ – the unit disk with a small bump of radius a removed (see Figure 2).

Fix a large $n\in\mathbb{N}$. Let now for $0\leq l<2^n$, and let $\Omega_l=e^{2\pi il/2^n}\Lambda_{2^{-10n}}$ be the rotated domain $\Lambda_{2^{-10n}}$. For a set $L=\{l_1,l_2,\ldots,l_k\}$ with all $0\leq l_1<l_2\ldots l_k<2^n$, let $\Omega_L=\Omega_{l_1}\cap\Omega_{l_2}\cap\ldots\Omega_{l_k}$. Thus Ω_L is the unit disk with k relatively "spread out" bumps removed.

Theorem 3.1. *For large enough n,*

$$\left|r(\Omega_L,0)-1+k2^{-20n-1}\right|<\frac{1}{10}2^{-20n}.$$

To prove Theorem 3.1, we estimate the conformal radius of Λ_a for an arbitrary a.

Lemma 3.2. *The conformal radius of Λ_a is equal to* $\dfrac{2-2a}{2-2a+a^2}$.

As a consequence we get that, for large n,

$$|r(\Lambda_{2^{-10n}})-1+2^{-20n-1}|<2^{-30n+2} \tag{3.1}$$

Proof of Lemma 3.2. Let $\mathbb{P}=\mathbb{C}\setminus\{\operatorname{Im}z=0,\ \operatorname{Re}z\leq 0\}$ be the complex plane with the negative real axis removed.

The function

$$\chi(z)=\left(\frac{1-z}{1+z}\right)^2$$

maps $\mathbb{D}$ conformally onto $\mathbb{P}$, $\chi(0)=1$. It also maps Λ_a onto $\Lambda_b'=\mathbb{P}\setminus\{|z|\leq b\}$, where $b=\left(\dfrac{a}{2-a}\right)^2$.

Observe also that

$$h(z)=\frac{z+b^2/z-2b}{(1-b)^2}$$

maps Λ_b' conformally onto $\mathbb{P}$, with $h(1)=1$, $h'(1)=\dfrac{1+b}{1-b}=\dfrac{2-2a+a^2}{2-2a}$.

Thus the map $\phi_0(z) = \chi^{-1} \circ h^{-1} \circ \chi(z)$ maps $\mathbb{D}$ conformally onto Λ_a, and the conformal radius of Λ_a is equal to

$$r(\Lambda_a) = 1/h'(1) = \frac{2-2a}{2-2a+a^2}.$$

□

For a set L, let ϕ_L be the conformal map of $\mathbb{D}$ onto Ω_L with $\phi_L(0) = 0$, $\phi'_L(0) > 0$ (ϕ_L is the inverse of the uniformization map). Let $L' = (l_2, l_3, \ldots, l_k)$ be the set L with the first element removed. Let $g(z) = \phi_{L'}^{-1} \circ \phi_L(z)$ be the conformal map of $\mathbb{D}$ onto

$$\Gamma = \mathbb{D} \setminus \phi_{L'}^{-1}\left(\{|z| < 1,\ |z - e^{2\pi i l_1/2^n}| \leq 2^{-10n}\}\right).$$

Let us also introduce two domains

$$\Gamma_+ = \mathbb{D} \setminus \{z\ :\ |w - z| \leq 2^{-10n}(1 - 2^{-4n})\}$$

and

$$\Gamma_- = \mathbb{D} \setminus \{z\ :\ |w - z| \leq 2^{-10n}(1 + 2^{-4n})\},$$

where $w = \phi_{L'}^{-1}(1)$.

We will use the following property of Γ

Lemma 3.3.

$$\Gamma_- \subset \Gamma \subset \Gamma_+.$$

Let us first show how to derive Theorem 3.1 from Lemma 3.3.
By Lemma 3.3 and Lemma 1.3 (monotonicity of conformal radius),

$$r(\Gamma_-) \leq g'(0) = r(\Gamma) \leq r(\Gamma_+)$$

Now Lemma 3.2 implies that for large n

$$|r(\Gamma_-) - 1 + 2^{-20n-1}| < 2^{-24n+2}, \quad |r(\Gamma_+) - 1 + 2^{-20n-1}| < 2^{-24n+2},$$

and thus

$$|g'(0) - 1 + 2^{-20n-1}| < 2^{-24n+2}.$$

Note now that $\phi_L(z) = \phi_{L'} \circ g(z)$, so $r(\Omega_L) = g'(0) r(\Omega_{L'})$. The Theorem easily follows from this relation by induction on the size of L.
So to establish Theorem 3.1, it is enough to prove Lemma 3.3.

Proof of Lemma 3.3. Without loss of generality we can assume that $l_1 = 0$.

Let $\Upsilon = \mathbb{D} \cap \{|z - 2^{-7n+1}| < 1 - 2^{-7n}\}$. Note that $\Upsilon \subset \Omega_{L'}$, since $0 \notin L'$. Let ψ be the conformal map of $\mathbb{D}$ onto Υ with $\psi(0) = 0$, $\psi'(0) > 0$.

Let K be the arc $[1, e^{\pi i 2^{-7n}}]$. Note that $K \subset \partial\Upsilon \cap \partial\Omega_{L'}$. Let $K' = \phi_{L'}^{-1}(K)$. The normalized length of K', $|K'|$, is the harmonic measure of K in $\Omega_{L'}$ evaluated at zero. By monotonicity of harmonic measure (Lemma 1.6) it is bounded above by $2^{-7n-1} = |K|$ and below by the harmonic measure of K in Υ evaluated at zero.

So

$$2^{-7n-1} \geq |K'| \geq |\psi^{-1}(K)| \geq 2^{-7n-1}(1 - 2^{-5n}) \tag{3.2}$$

The same estimate applies to $K'' = \phi_{L'}^{-1}([e^{-\pi i 2^{-7n}}, 1])$.

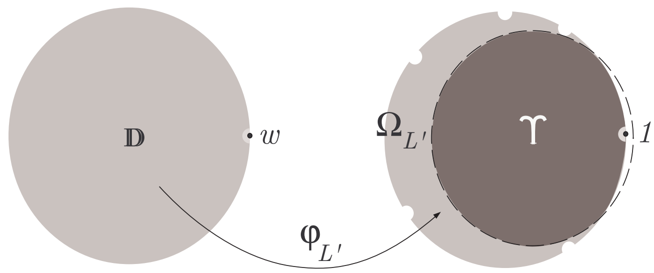

FIGURE 3. The map $\phi_{L'}$ and domain Υ

.

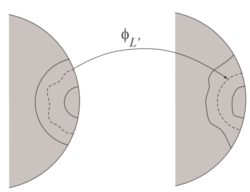

FIGURE 4. $\phi_{L'}$ in the neighborhood of w

By the Reflection Principle 1.4, the map $\phi_{L'}$ can be extended to a map G of the whole disk $\{|z-w|<2^{-7n}(1-2^{-5n})\}$ with $G(w)=1$.

Using the fact that $\Upsilon \subset \Omega_{L'}$, we obtain

$$|G'(w)| = |\phi_{L'}(w)| > |\psi'(w)| > 1-2^{-7n} \tag{3.3}$$

Now we can use the Distortion Theorem 1.7 applied to the disk $\{|z-w|<2^{-7n}(1-2^{-5n})\}$ to see that

$$G(\{z\ :\ |w-z|\leq 2^{-10n}(1-2^{-4n})\}) \subset \{|z-1|<2^{-10n}\}$$

and

$$\{|z-1|<2^{-10n}\} \subset G(\{z\ :\ |w-z|\leq 2^{-10n}(1+2^{-4n})\})$$

But this is precisely the statement of the lemma.

□

Now we are in the position to prove Theorems 1.10 and 1.11.

Proof of Theorem 1.10. For a propositional formula Φ with n variables, let $L \subset \{0, 1, \ldots, 2^n - 1\}$ be the set of numbers corresponding to its satisfying instances. Then the boundary of Ω_L is computable in linear time, given the access to Φ. Theorem 3.1 now implies that using $r(\Omega_L, 0)$ we can evaluate $|L| = k$, and solve the $\#SAT$ problem on Φ, which is exactly Theorem 1.10. □

Proof of Theorem 1.11. Suppose that we are given a string s of $n = 2^k$ zeros and ones. We can view it as a set $L \subset \{0, 1, \ldots, 2^k - 1\}$. Ω_L can be obtained from L by a trivial one-layered circuit with just NOT gates. Theorem 3.1 implies that using $r(\Omega_L, 0)$ with $2^{-O(k)}$ precision, we can evaluate $|L|$ and solve the MAJ_n problem on s, which is exactly Theorem 1.11. □

References

[Ahl] Ahlfors, L. V., Complex analysis. An introduction to the theory of analytic functions of one complex variable, Third edition, International Series in Pure and Applied Mathematics, McGraw-Hill Book Co., New York, 1978.

[BB] Bishop, E. and Bridges, D., Constructive analysis, Grundlehren der Mathematischen Wissenschaften, 279. Springer-Verlag, Berlin, 1985.

[BW] Brattka, V., Weihrauch, K., Computability of Subsets of Euclidean Space I: Closed and Compact Subsets, Theoretical Computer Science, 219, 1999, 65–93.

[Brav] Braverman, M., On the Complexity of Real Functions, Proceedings of 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), 23-25 October 2005, Pittsburgh, PA, USA, 2005, 155–164.

[Cheng] Cheng, H., A constructive Riemann mapping theorem, Pacific J. Math. 44 (1973), 435–454.

[Chou] Chou, A. W., Some complexity issues in complex analysis, in Proc. Workshop on Computability and complexity in Analysis, Eds. K.-I. Ko, K. Weihrauch, Informatik-Berichte, vol. 190, 1995, 91-98.

[Dur] Duren, P. L., Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York, 1983.

[FSS] Furst, M., Saxe, J. B., and Sipser, M., Parity, circuits, and the polynomial-time hierarchy, Math. Systems Theory 17 (1984), no. 1, 13–27.

[GM] Garnett, J. B. and Marshall, D. E., Harmonic measure, New Mathematical Monographs, 2. Cambridge University Press, Cambridge, 2005.

[Hert] Hertling, P., An effective Riemann mapping theorem. (English summary), Computability and complexity in analysis (Castle Dagstuhl, 1997), Theoret. Comput. Sci. 219 (1999), no. 1-2, 225–265.

[Koebe] Koebe, P., *Über eine neue Methode der konformen Abbildung und Uniformisierung*, Nachr. Königl. Ges. Wiss. Göttingen, Math. Phys. Kl., 1912, 844-848.

[Laa] Laasonen, P., On the behavior of the solution of the Dirichlet problem at analytic corners, Ann. Acad. Sci. Fenn. Ser. A. I. 1957 (1957), no. 241, 13 pp. 35.0X

[Mar] Marshall, D. E., "Zipper", Fortran programs for numerical computation of conformal maps, and C programs for X11 graphics display of the maps, Preprint, available from *http://www.math.washington.edu/˜marshall*

[MR] D.E. Marshall, S. Rohde, Convergence of the Zipper algorithm for conformal mapping, Preprint, available from *http://www.math.washington.edu/˜rohde*

[Nis] Nisan, N., RL $\subseteq$ SC, Comput. Complexity 4 (1994), no. 1, 1–11.

[Papa] Papadimitriou, C. H., Computational complexity, Addison-Wesley Publishing Company, Reading, MA, 1994

[Pom] Pommerenke, C., Boundary behaviour of conformal maps, Grundlehren der Mathematischen Wissenschaften, 299, Springer-Verlag, Berlin, 1992.

[RW] Rettinger, R., Weihrauch, K., The Computational Complexity of Some Julia Sets, Proceedings of the 35th Annual ACM Symposium on Theory of Computing (STOC), June 9-11, 2003, San Diego, CA, USA, 2003, 177–185.

[Sip] Sipser, M., Introduction to the Theory of Computation, Second Edition, PIndependence, Kentucky, Course Technology Ptr., 2005

[Spi] Spitzer, F., Principles of random walks, Second edition, Graduate Texts in Mathematics, Vol. 34. Springer-Verlag, New York-Heidelberg, 1976.

[Thu] Thurston, W. P., Three-dimensional geometry and topology, Vol. 1, Edited by Silvio Levy, Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997.

[Wei] Weihrauch, K., Computable analysis, (English summary) An introduction, Texts in Theoretical Computer Science, An EATCS Series, Springer-Verlag, Berlin, 2000.

[Zhou] Zhou, Q., Computable real-valued functions on recursive open and closed subsets of Euclidean space, Math. Logic. Quart. 42 (1996), 379-409.