PHASE TRANSITION FOR THE UNIVERSAL BOUNDS ON THE INTEGRAL MEANS SPECTRUM

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Abstract. In this paper we generalize Beurling's estimate on the rate of decay of harmonic measure near a boundary point with given rotation. The generalization allows us to establish the existence of phase transition for the universal bounds on the integral means mixed spectrum of bounded simply connected domains.

1. Introduction

In what follows, $\Omega \subset \mathbb{C}$ is a simply connected domain, w_0 is an interior point of Ω, and $\phi : \mathbb{D} \to \Omega$ is the Riemann map with $\phi(0) = w_0$, $\phi'(0) > 0$. $\omega(E)$ will denote the harmonic measure of $E \subset \partial\Omega$ evaluated at w_0. For a point $v \in \mathbb{C}$, $B(v, \delta)$ will denote the closed ball centered at v of radius δ.

It is known (see [GM08]), that for any $v \in \partial\Omega$, the harmonic measure of a small ball, $\omega(B(v, \delta))$ decays with the rate at least $\delta^{1/2}$ when $\delta \to 0$. It follows from Makarov's dimension theorem (see [Mak85]) that a.e. by harmonic measure, $\omega(B(v, \delta))$ decays like δ. More precisely, for ω - a.e. $v \in \partial\Omega$,

$$\lim_{\delta\to 0} \frac{\log \omega(B(v, \delta))}{\log \delta} = 1.$$

These results can be significantly refined using Multifractal Analysis (see, for example, [Fal90]). Roughly speaking, *dimension spectrum of* ω, $f(\alpha)$ is defined as the dimension of the set of points with the rate of decay (or *local dimension*) of harmonic measure equal to α. This "naive" version of the definition only makes sense for "nice" domains Ω, such as Carleson fractals and Julia sets (see [Mak98, Zin00]).

For general domains, there are a few ways to make the notion of the dimension spectrum useful. We will use the following version of the spectrum (see [Mak98]).

Definition 1. The *Minkowski dimension spectrum of a simply connected domain* Ω is defined as

$$f(\alpha) = f_\Omega(\alpha) = \lim_{\eta\to 0} \limsup_{\delta\to 0} \frac{\log N(\delta, \alpha, \eta)}{\log \frac{1}{\delta}},$$

where $N(\delta, \alpha, \eta)$ is the maximum number of points $v_n \in \partial\Omega$, such that $|v_j - v_k| > 2\delta$ if $j \neq k$ and $\delta^{\alpha+\eta} \le \omega(B(v_n, \delta)) \le \delta^{\alpha-\eta}$.

Note that for each α either $f_\Omega(\alpha) \ge 0$ or $f_\Omega(\alpha) = -\infty$. The latter occurs if for all small δ and η there are no disks with the prescribed amount of harmonic measure.

The above-mentioned upper estimate of harmonic measure implies that $f(\alpha) = -\infty$ whenever $\alpha < 1/2$. Makarov's theorem implies that $f(1) = 1$ and $f(\alpha) < \alpha$ if $\alpha \neq 1$.

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Closely related to the dimension spectrum is one of the central objects of the Theory of Conformal maps, the *Integral Means Spectrum*, defined as

$$\beta_\Omega(t) = \beta_\phi(t) = \limsup_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^t(\zeta)||d\zeta|}{\log \frac{1}{1-r}}. \tag{1}$$

The value of the spectrum is independent of the choice of the Riemann map $\phi : \mathbb{D} \to \Omega$. We refer to [Mak98] and [Pom92] for the discussion of the spectrum and its properties.

Integral means spectrum of a domain provide an upper bound on the dimension spectrum of harmonic measure in the form of the following Legendre-type transform (see [Mak98]):

$$f_\Omega(\alpha) \leq \inf_t \left(\alpha(\beta_\Omega(t) - t + 1) + t\right) \tag{2}$$

$$\beta_\Omega(t) \geq \sup_\alpha \left(\frac{f_\Omega(\alpha) - t}{\alpha} + t - 1\right). \tag{3}$$

For Carleson fractals and Julia sets, these inequalities become equalities. Using this observation and Fractal Approximation, one can see that the questions about sharp upper bounds on the dimension spectrum and integral means spectrum can be studied simultaneously. More precisely, the *Universal Integral Means Spectrum*

$$B_b(t) = \sup_{\text{Bounded simply connected } \Omega} \beta_\Omega(t)$$

and the *Universal Dimension Spectrum* as

$$F(\alpha) = \sup_{\text{simply connected } \Omega} f_\Omega(\alpha)$$

are related by the Legendre-type transforms (2), (3) with the equalities instead of inequalities (see [Mak98]):

$$F(\alpha) = \inf_t \left(\alpha(B_b(t) - t + 1) + t\right) \tag{4}$$

$$B_b(t) = \sup_\alpha \left(\frac{F(\alpha) - t}{\alpha} + t - 1\right). \tag{5}$$

A lot of classical questions about the boundary behavior of conformal maps and the fine properties of harmonic measure can be reformulated in terms of Universal Integral Means Spectrum or, equivalently, Universal Dimension Spectrum. For example, the celebrated Brennan conjecture can be restated as $B_b(-2) = 2$.

The strongest conjecture about the value of $B_b(t)$ was made by by Ph. Kraetzer (see [Kra96]):

Conjecture 1.

$$B_b(t) = \begin{cases} \frac{t^2}{4}, & |t| \leq 2, \\ |t| - 1, & |t| > 2 \end{cases}.$$

Equivalently,

$$F(\alpha) = 2 - \frac{1}{\alpha}$$

whenever $\alpha \geq 1/2$

We refer to [BS05] and [HS08] for the survey of the recent progress related to Kraetser conjecture.

One of the most interesting features of the conjectured behavior of $B_b(t)$ is the existence of the phase transition phenomenon: the function becomes linear when $|t| > 2$. Since $B_b(t) \geq |t| - 1$ and $B_b(t)$ is convex ([Pom92]), the existence of above-mentioned phase transition is equivalent to the equality $B_b(t) = t - 1$ for some $t < 0$ (the existence of the phase transition for $t > 0$ follows from the easy identity $B_b(2) = 1$). This property was established by Carleson and Makarov in [CM94] using the estimates on the Universal Dimension Spectrum near $\alpha = 1/2$ and the Legendre-type relations (4), (5).

Various modern approaches to the verification of Conjecture 1 make use of a slight generalization of the integral means spectrum. Namely, one allows the exponent to be a complex number.

Definition 2. The *Integral Mixed Spectrum* of a simply connected domain Ω with the Riemann map $\phi : \mathbb{D} \to \Omega$ is

$$m_\Omega(z) = m_\phi(z) = \limsup_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^{z}(\zeta)| d|\zeta|}{\log \frac{1}{1-r}}$$

where $z \in \mathbb{C}$.

Since $\phi'(\zeta) \neq 0$ in $\mathbb{D}$, the complex powers of $\phi'(\zeta)$ are well defined, and the rate of growth does not depend on the branch chosen. m_Ω depends not only on the boundary behavior of $|\phi'(\zeta)|$, but also on the growth of $\arg \phi'(\zeta)$, or the rotation of the image of the radius.

Universal Integral Mixed Spectrum is defined the same as the Integral Means counterpart:

$$M_b(z) = \sup_{\text{Bounded simply connected } \Omega} m_\Omega(z).$$

In [BP87], Becker and Pommerenke extended the Brennan conjecture to the complex case, asking whether $M_b(z) = 1$ whenever $|z| = 2$. As in the real-exponent case, since $M_b(z) \geq |z| - 1$ and $M_b(z)$ convex, this conjecture would imply that $M_b(z) = |z| - 1$ whenever $|z| \geq 2$. In this paper, we establish a weak form of the last conjecture.

Theorem 1. *For each θ, $0 \leq \theta < 2\pi$ there exists $T_\theta > 0$ such that $M_b(te^{i\theta}) = t - 1$ for $t \geq T_\theta$.*

Roughly speaking, Theorem 1 states that for $t \geq T_\theta$, the extremal growth of the mean values of the complex powers of the derivatives occurs near a single spiral. Since it is known that $M_b(0) = 0 > 0 - 1$, this is not the case for small values of t. Furthermore, one can show that for $|z| < 2$ we have $M_b(z) > |z| - 1$. Consequently, we have a phase transition for the behavior of $M_b(z)$ along some curve surrounding zero. Becker-Pommerenke conjecture implies that this critical curve is the circle $|z| = 2$. We conjecture that the following generalization of Kraetzer conjecture actually holds:

Conjecture 2.

$$M_b(z) = \begin{cases} \frac{|z|^2}{4}, & |z| \leq 2, \\ |z| - 1, & |z| > 2 \end{cases}.$$

The smooth phase transition at $z = 2$, predicted by this conjecture, is shown to exist by Jones and Makarov in [JM95] for the real values of z, and by Baranov and Hedenmalm in [BH08] for the complex values of z.

The geometric counterpart of the Integral Mixed Spectrum is the Dimension Mixed Spectrum. To define it, we will need to introduce a notion of *rotation near a boundary point*, which essentially corresponds to the growth of $\arg \phi'$.

Definition 3. Let $v \in \partial\Omega$, $\delta > 0$, Ω_δ be the connected component of $\Omega \setminus B(v, 2\delta)$ containing w_0. Define the *rotation of the domain Ω near v at the distance δ* as

$$\rho(v, \delta) = \exp\Big(\inf_{w \in \partial\Omega_\delta \cap B(v, 2\delta)} \arg(w - v)\Big),$$

where the branch of the function $g(w) = \arg(w - v)$ is selected so that $-\pi < \arg(w_0 - v) \leq \pi$.

Since Ω is simply-connected, the branch of $g(w)$ is well-defined in Ω. In other words, $\rho(v, \delta)$ measures how many times a curve should rotate around point v before it first gets 2δ-close to v within Ω. An easy topological observation ([Bin]) shows that

(6) For any two points $w_1,\ w_2 \in \partial\Omega_\delta \cap B(v, 2\delta),$ we have $|\arg(w_1 - v) - \arg(w_2 - v)| \leq 2\pi$

Let us also note that we consider the exponent of the argument in Definition 3 to scale the rotation the same way as the harmonic measure of the corresponding disc.

At this point we can introduce a two parameter generalization of the dimension spectrum, the *Dimension Mixed Spectrum* (See [Bin] for further motivation and variants of the definition).

Definition 4. The *Dimension Mixed Spectrum of a simply connected domain* Ω is

$$f_\Omega(\alpha,\gamma) = \lim_{\eta\to 0}\limsup_{\delta\to 0}\frac{\log N(\delta,\alpha,\gamma,\eta)}{\log\frac{1}{\delta}},$$

where $N(\delta,\alpha,\gamma,\eta)$ is the maximum number of points $v_n \in \partial\Omega$, such that $|v_j - v_k| > 2\delta$ if $j \neq k$, $\delta^{\alpha+\eta} \leq \omega(B(v_n,\delta)) \leq \delta^{\alpha-\eta}$, and $\delta^{\gamma+\eta} \leq \rho(v_n,\delta) \leq \delta^{\gamma-\eta}$.

Intuitively, $f(\alpha,\gamma)$ measures the dimension of the set of point with the local dimension of harmonic measure equal to α and the rate of boundary rotation is equal to γ. As in non-rotational case, $f_\Omega(\alpha,\gamma)$ is either non-negative or equals to $-\infty$.

The following geometric observation was essentially made by Beurling in [Beu89]. It generalizes the lower bound of $\delta^{1/2}$ on the rate of decay of harmonic measure.

Lemma 1. *Let $f_\Omega(\alpha,\gamma)\geq 0$ for some simply connected Ω. Then $\alpha \geq \frac{1}{2}+\frac{\gamma^2}{2}$. In other words, if $\alpha+\eta < \frac{1+(\gamma-\eta)^2}{2}$, then there is no $v\in\partial\Omega$ and $\delta>0$ such that*

$$\omega(B(v,\delta)) \geq \delta^{\alpha+\eta} \text{ and } \delta^{\gamma+\eta} \leq \rho(v,\delta) \leq \delta^{\gamma-\eta}.$$

The characterization is sharp: for any $\alpha \geq \frac{1}{2}+\frac{\gamma^2}{2}$ there exists a bounded domain Ω with $f_\Omega(\alpha,\gamma)\geq 0$.

Remark. Lemma 1 is equivalent to the well-known Pommerenke-Ýoccoz-Levin inequality.

As shown in [Bin], the Legendre-type relations between the Integral Means Spectrum and Dimension Spectrum (2), (3) generalize to the rotational case. Namely, for any simply-connected domain Ω,

$$f_\Omega(\alpha,\gamma) \leq \inf_z\left(\alpha m_\Omega(z) + (1-\alpha)\operatorname{Re}(z) - \gamma\operatorname{Im}(z)+\alpha\right) \tag{7}$$

$$m_\Omega(z) \geq \sup_{\alpha,\gamma}\left(\frac{f_\Omega(\alpha,\gamma)-(1-\alpha)\operatorname{Re}(z)+\gamma\operatorname{Im}(z)-\alpha}{\alpha}\right). \tag{8}$$

Using the Fractal Approximation, one can also generalize (4), (5) ([Bin]):

$$M_b(z) = \sup_{\alpha,\gamma}\left(\frac{F(\alpha,\gamma)-(1-\alpha)\operatorname{Re}(z)+\gamma\operatorname{Im}(z)-\alpha}{\alpha}\right) \tag{9}$$

$$F(\alpha,\gamma) = \inf_z\left(\alpha M_b(z) + (1-\alpha)\operatorname{Re}(z) - \gamma\operatorname{Im}(z)+\alpha\right). \tag{10}$$

where $F(\alpha,\gamma)$ denote the *Universal Dimension Mixed Spectrum*

$$F(\alpha,\gamma) = \sup_{\text{simply connected }\Omega} f_\Omega(\alpha,\gamma)$$

It will be more convinient for us to work with Dimension Spectrum. More specifically, we will establish the following generalization of the Beurling's estimate (Lemma 1), which implies Theorem 1:

Theorem 2. $F(\alpha,\gamma)\leq C\left(\alpha-\frac{1+\gamma^2}{2}\right)$ *whenever* $\alpha\geq\frac{1+\gamma^2}{2}$.

Let us note that $C=2$ in this inequality is equivalent to the Becker-Pommerenke conjecture. Conjecture 2 can be restated in terms of the Dimension Mixed Spectrum as

Conjecture 3.

$$F(\alpha,\gamma) = 2 - \frac{1+\gamma^2}{\alpha}$$

whenever $\alpha \geq \frac{1+\gamma^2}{2}$.

The rest of the paper is organized as follows. First, for the sake of completeness, as well as to motivate our further arguments, we give the proof of Lemma 1. Than we explain the equivalence of Theorem 1 and Theorem 2. Finally we provide a refinement of Lemma 1 and a combinatorial construction necessary for the proof of Theorem 2.

2. Estimates on the harmonic measure in the presence of large rotation.

The crucial technical tool for our proofs of both Lemma 1 and the main theorems is the *extremal distance*.

Definition 5. Let Ω be a domain, and $E,\ F \subset \overline{\Omega}$. For a non-negative continuous function ρ on Ω, let

$$A(\rho) = \iint_\Omega \rho^2\,dx\,dy$$

and

$$L(\rho) = \inf_\sigma \int_\sigma \rho\,ds,$$

where the infimum is taken over all rectifiable arcs σ in Ω joining E to F. *The extremal distance from E to F within* Ω is defined as

$$\lambda_\Omega(E,F) = \sup_\rho \frac{L^2(\rho)}{A(\rho)},$$

where the supremum now is taken over all ρ with $A(\rho) > 0$.

We refer to, say, [GM08] for the discussion of the properties of the extremal distance and the connections with harmonic measure. We will make use of the conformal invariance of extremal distance, the sub additivity, and the fact that to estimate the extremal distance from bellow one just need to produce a "nice" metric ρ. Some finer properties of the extremal distance will be introduced as they are required.

Proof or the Lemma 1. Let us note that the use of symmetry and equation (6) imply that $F(\alpha,\gamma) = F(\alpha,-\gamma)$. So without loss of generality we can assume that γ is positive. Let us fix a simply connected domain Ω , a point $v \in \partial\Omega$, a point $w_0 \in \Omega$. Let L be the arc of the circle centered at v of the radius $R = |w_0 - v|$ which forms a crosscut of Ω containing w_0.

Let now $B(v,\delta)$ be a ball with the harmonic measure $\omega(B(v,\delta)) > \delta^{\alpha+\eta}$ and the rotation $\rho(v,\delta) < \delta^{\gamma-\eta}$. Then (see [Mak87], Corollary 1.4) there exists an arc $l \subset \{w\ :\ |w-v| = 2\delta\} \cap \Omega$ such that

$$\lambda_\Omega(l,L) \leq \left(\frac{\alpha+\eta}{\pi} + \epsilon(\delta)\right)|\log\delta|, \tag{11}$$

where $\epsilon(\delta) \to 0$ when $\delta \to 0$.

Let us consider a branch $g_v(w)$ of the function $\log(w-v)$ in the domain Ω. To be consistent with the definition of the rotation, let us normalize g_v to take the principal value for $w = w_0$. By the conformal invariance of extremal distance, $\lambda_\Omega(l,L)$ is the same as the extremal distance between $g_v(L) \subset \{u\ :\ \mathrm{Im}(u) = \log R\}$ and $g_v(l)$.

To obtain a lower estimate on $\lambda_\Omega(l,L)$, we consider the domain

$$\Omega' = g_v(\Omega_\delta \cap \Omega_0),$$

where Ω_0 is the connected component of the set $\Omega \setminus L$ containing the arc l.

In the standard Euclidean metric, the area of the domain is at most $2\pi \log \frac{R}{2\delta}$, since the projection of the Ω' to the real axis has the length $\log \frac{R}{2\delta}$, and the length of each vertical crosscut is at most 2π. On the other hand, since the rotation is at least $(\gamma-\eta)|\log 2\delta|$, the Euclidean distance between $g(l)$ and $g(L)$ is at least $\sqrt{1+(\gamma-\eta)^2}\log\frac{R}{2\delta} - 4\pi$. Thus, comparing the extremal distance to the quantity obtained for the Euclidean metric on Ω', we obtain the estimate

$$\lambda_\Omega(l,L) \geq \frac{1}{\pi}\left(\frac{1}{2}+\frac{(\gamma-\eta)^2}{2}-\epsilon'(\delta)\right)\left(\log\left(\frac{R}{2\delta}\right)\right), \tag{12}$$

where $\epsilon'(\delta)\to 0$ when $\delta\to 0$.

Combination of (11) and (12) gives the inequality

$$\frac{1}{\pi}\left(\frac{1}{2}+\frac{(\gamma-\eta)^2}{2}-\epsilon'(\delta)\right)\left(\log\left(\frac{R}{2\delta}\right)\right) \leq \left(\frac{\alpha+\eta}{\pi}+\epsilon(\delta)\right)|\log\delta|,$$

which implies the first assertion of the Lemma if δ is small enough.

To prove the exactness, let us fix γ. Take $t = \arctan\gamma$ and consider the domain $h(\mathbb{D})$, where $h(z) = (1-z)^{\lambda\cos t e^{it}}$. The map h is bounded and univalent in $\mathbb{D}$ for $0<\lambda\leq 2$. Near the point $v = h(1) = 0$ the local dimension of harmonic measure of $h(\mathbb{D})$ is equal to $\alpha = \frac{1}{\lambda\cos^2 t} = \frac{1+\gamma^2}{\lambda}$ and the local rotation rate is equal to γ. □

Proof that Theorem 2 is equivalent to Theorem 1. Let us first suppose that Theorem 2 holds. Observe that for any θ, $(\alpha-1)\cos\theta+\gamma\sin\theta \leq \sqrt{(\alpha-1)^2+\gamma^2}$. Thus we have

$$F(\alpha,\gamma) \leq C(\alpha-(\alpha-1)\cos\theta-\gamma\sin\theta)$$

for any θ. So, by (9),

$$M_b(te^{i\theta}) \leq \sup_{\alpha,\gamma}(C-t)\left(1-\left(1-\frac{1}{\alpha}\right)\cos\theta-\frac{\gamma}{\alpha}\sin\theta\right)+t-1 = t-1$$

for $t = C =: T$. Thus, for $|z| = T$, $M_b(z)\leq |z|-1$. Since M_b is a convex increasing in every direction function and $M_b(z)\geq |z|-1$, this implies that $M_b(z) = |z|-1$ as soon as $|z|\geq T$.

On the other hand, if Theorem 1 holds, then by (10)

$$\begin{aligned}F(\alpha,\gamma) = \inf_z\left(\alpha M_b(z)+(1-\alpha)\operatorname{Re}(z)-\gamma\operatorname{Im}(z)+\alpha\right) \leq \\ \inf_{\theta,\ t\geq T}\left(\alpha(t-1)+(1-\alpha)t\cos\theta-\gamma t\sin\theta+\alpha\right) = T(\alpha-\sqrt{(\alpha-1)^2+\gamma^2}).\end{aligned}$$

□

It will be more convenient for us to deal with the dimension spectrum, so we prove Theorem 2. Let us note that because of the symmetry property of the universal spectrum, it is enough to consider only nonnegative γ. The case $\gamma = 0$ immediately follows from Theorem 1 of [CM94], which essentially states that $\sup_\gamma F(\alpha,\gamma)\leq C(\alpha-\frac{1}{2})$. Thus we can consider only the case $\gamma>0$.

Let us fix $\gamma>0$ and $\alpha = \frac{\gamma^2}{2}+\frac{1}{2}+\epsilon,\ \epsilon>0$. Let us also denote $\mu = \sqrt{1+\gamma^2}$. It will be convenient to normalize Ω by the condition $\operatorname{diam}\Omega = 10$.

As in the proof of Lemma 1, for a point $v\in\partial\Omega$ let $g_v(w)$ be a branch of $\log(w-v)$. Let us also fix a large constant A. Let L'_n be the line $x = -\gamma y - nA - \log\mu$ and P'_n be the half plane $x\geq -\gamma y - nA-\log\mu$.

We denote by $P_n(v)$ the connected component of $g_v(\Omega)\cap P'_n$ containing $g_v(w_0)$, $R_n(v) = P_{n+1}(v)\setminus P_n(v)$, and $L_n(v) = \partial R_n(v)\cap L'_n$. We denote the area of $R_n(v)\cap\left(P'_{n+1}\setminus P'_n\right)$ by $S_n(v)$.

Lemma 2. *Let $\tilde\lambda(n,v) = \inf \lambda_{R_n(v)}(l_n, l_{n+1})$, where the infimum is taken over all the connected subsets l_n and l_{n+1} of $L_n(v)$ and $L_{n+1}(v)$ correspondingly. Then $\tilde\lambda(n,v) \geq \frac{A^2}{S_n(v)\mu^2}$. Equality is reached only when $R_n(v)$ is a rectangle with the sides parallel to the lines $y = \gamma x$ and $x = -\gamma y$.*

Proof. To obtain this estimate, we put the standard Euclidean metric on $R_n(v) \cap \left(P'_{n+1} \setminus P'_n\right)$ and 0 on the rest of $R_n(v)$, that is on $R_n(v) \cap P'_n$. Then, since the Euclidean length of intersection of any curve joining L_{n-1} and L_n with $P'_{n+1} \setminus P'_n$ is at least $\frac{A}{\mu}$, we have

$$\tilde\lambda(n,v) \geq \frac{A^2}{S_n(v)\mu^2} \,.$$

The equality in the first inequality is only achieved when this metric is extremal, and thus $R_n(v)$ is a rectangle as in the statement of the lemma. □

Let $T_n(v) = \max(S_n(v) - \frac{8A\pi}{\mu^2}, 0)$, the "excess area" of $R_n(v)$. Notice that we need to subtract four times the area of $R_n(v)$ for the logarithmic double spiral with parameter γ. Define $\tilde S_n(v) := \frac{A(T_n(v))^2}{S_n^2}$. For the future reference, note that $\tilde S_n(v) \leq \eta^2$ for some positive η if and only if $S_n(v) \leq \frac{8\pi A}{(1-\frac{\eta}{A})\mu^2}$.

Let $Y_n(v) = \tilde\lambda(n,v) - \frac{A^2}{S_n(v)\mu^2} + \tilde{\mathrm{S}}_n(v)$ denote the "excess" of the extremal distance in $R_n(v)$. Note that by Lemma 2 and the preceding observation, $Y_n(v) = 0$ if and only if $R_n(v)$ is a rectangle with the sides parallel to the lines $y = \gamma x$ and $x = -\gamma y$, one of the sides equal A and the other is at most $\frac{8\pi}{\mu}$. Lemma 4 gives the quantitative version of this characteristic.

The following three technical lemmas allow us to estimate the number of non intersecting balls with fixed rotation and large harmonic measure.

Lemma 3. *For each $\epsilon > 0$ and $R = \mathrm{dist}(w_0, \partial\Omega)$, there exist $M = M(R)$ and $N = N(\epsilon, R)$ and, such that the following holds. Let $v \in \partial\Omega$, $\frac{1}{2}\exp(-\frac{A}{\mu^2}(n+M+1) - 2\pi\gamma\mu) < \delta \leq \frac{1}{2}\exp(-\frac{A}{\mu^2}(n+M) - 2\pi\gamma\mu)$ for some $n > N$, and $\rho(v,\delta) \leq \delta^\gamma$ and $\omega(B(v,\delta)) \geq \delta^\alpha$. Then*

$$\sum_{j=1}^{n} Y_{j+M}(z) \leq C\frac{A}{\mu^2} n\epsilon$$

for some absolute constant $C > 0$. In addition, $v + \exp\left(\cup_{j=M}^{M+n} R_j(v)\right) \cap B(v,\delta) = \emptyset$.

Proof. First, we pick M so that $R > \frac{1}{\mu}\exp(-MA + 2\pi)$. This normalization guarantees that $g_v(w_0) \in P'_M$.

As in the proof of Lemma 1, we can use Corollary 1.4 in [Mak87] to see that since $\omega(B(v,2\delta)) \geq \delta^\alpha$, there exist two arcs $l \subset \Omega \cap \partial B(v,\delta)$ and $L \subset \Omega \cap \partial B(v,R)$ with the extremal distance between them no greater than $\frac{1}{\pi}\alpha \log\frac{R}{\delta}$.

Because $\rho(v,\delta) \leq \delta^\gamma$ and because of our normalization of Ω, we have

$$\cup_{j=1}^{n} R_{j+M}(v) \subset \left(g_v(\Omega) \cap \{x + iy \;:\; \log 2\delta \leq x \leq \log R\}\right). \tag{13}$$

The second assumption of the lemma is the immediate consequence of this inclusion.

Combined with the subadditivity property of extremal distance, (13) implies that

$$\sum_{j=1}^{n} \tilde\lambda(j+M, v) \leq \lambda_\Omega(l, L) \leq \frac{1}{\pi}\,\alpha\log\frac{R}{\delta} \leq \tag{14}$$

$$\frac{\alpha\log(10) + 2\gamma\alpha\mu + \alpha\frac{A}{\mu^2}(n+1)}{\pi} = \frac{An}{2\pi}\left(1 + \frac{2\epsilon}{\mu^2} + \mathbf{o}(1)\right).$$

The last equality holds because $\alpha = \mu^2/2 + \epsilon$.

On the other hand, the inclusion (13) also implies that

(15)
$$\sum_{j=M}^{M+n} S_j(v) \le \text{ Area of } (g_v(\Omega) \cap \{x+iy \ : \ \log 2\delta \le x \le R\}) = 2\pi(\log 10 + \frac{A}{\mu^2}(n+1) + 2\pi\gamma\mu) = 2\pi\frac{A}{\mu^2} n(1+\mathbf{o}(1)).$$

Let now $x_1 \ge x_2 \ge \cdots \ge x_n > 0$ be any n positive numbers with $x_1 + x_2 + \ldots x_n \le B$. Let also $m \le n/2$ be the number of x_k with $x_k \ge \frac{2B}{n}$. Then we obtain the following extension of the Aritmetic Mean - Harmonic Mean inequality:

$$\sum_{j=1}^{n} \frac{1}{x_j} \ge \frac{(\sum_{j=1}^{n} \frac{1}{x_j})(\sum_{j=1}^{n} x_j)}{B} = \frac{n^2 + 1/2\sum_{i\ne j}\frac{(x_i-x_j)^2}{x_i x_j}}{B} \ge \frac{n^2}{B} + \frac{n}{2B}\sum_{j=1}^{m} \frac{(x_j - \frac{2B}{n})^2}{x_j^2} \tag{16}$$

Using (16) and (15) for $x_j = S_{j+M}$ (reordered in decreasing order) and $B = 2\pi\frac{A}{\mu^2} n(1+\mathbf{o}(1)) - 2\pi\sum_{j=M}^{M+n} a_j(v)$, we get that

$$\sum_{j=1}^{n} \frac{A^2}{\mu^2 S_{j+M}(v)} \ge \frac{(nA)^2 + nA\sum_{j=1}^{n} \tilde{\mathrm{S}}_{j+M}(v)}{2\pi An(1+\mathbf{o}(1))} \ge \frac{1}{2\pi}\left(\frac{An}{\mu^2} + \sum_{j=1}^{n} \tilde{\mathrm{S}}_{j+M}(v)\right)(1+\mathbf{o}(1)) \tag{17}$$

Now to estimate $\sum_{j=M}^{n+M} Y_j(v) = \sum_{j=M}^{n+M}\left(\tilde{\lambda}(j,v) - \frac{A^2}{S_j(v)\mu^2} + \tilde{\mathrm{S}}_j(v) + a_j(v)\right)$, we just subtract the estimate (17) from (14). □

For the next two lemmas, we will assume that $\operatorname{diam}\Omega \ge 10$. We need to change the normalization here for the rescaling argument that we use in the proof of Lemma 5 to work. In the proof of Theorem 2, the estimate will still be applied to domains with $\operatorname{diam}\Omega = 10$.

Lemma 4. *There exist positive absolute constants $\sigma > 0$ and $\tau > 0$ such that if $v \in \partial\Omega$ and $Y_1(v) \le e^{-\sigma kA}$, then for any $v_1 \in (v + \exp(\partial R_1(v))) \cap \partial\Omega$, we have $Y_j(v_1) \ge \tau A$ if $2 \le j \le k$.*

Proof. Since $\operatorname{diam}\partial\Omega \ge 10$, the boundary $g_v(\partial\Omega)$ intersects both $L_1(v)$ and $L_2(v)$.

Let c denote the length of orthogonal projection $g_v(\partial\Omega) \cap R_1$ to L_1. Remark that $c = 0$ iff the equality in Lemma 2 is reached. Note that since $\tilde{\mathrm{S}}_1(v) \le e^{-\sigma A}$, we get that $S_1(v) \le \frac{10\pi A}{\mu^2}$ for large enough σ and A. Since the Euclidean distance between $L_n(v)$ and $L_{n+1}(v)$ is at least $\frac{A}{\mu}$, we can apply an estimate from [Jen70] to get

$$e^{-\sigma kA} \ge Y_1(v) \ge \text{const}\,\frac{(\mu c)^3}{A}. \tag{18}$$

So $\partial\Omega \cap (v + \exp(R_1(v)))$ lies at a distance at most const $\frac{A}{\mu}\exp^{-\frac{\sigma}{3}kA}$ from a logarithmic spiral with parameter γ. The distance is less then $\frac{1}{\mu}e^{-kA}$ if $\sigma > 3$ and A is large enough. It means that for $v_1 \in \partial\Omega \cap (v + \exp(R_1(v)))$, the boundary spiral-like set $v_1 + \exp(R_i(v_1))$ is contained in a strip of the width $\frac{1}{\mu}e^{-kA}$, so there is no rotation around v_1 at this scale.

At this moment we need to consider two cases. In the case $\mu \ge 4$, we use the fact that $S_i(v) \ge \frac{3\pi}{4}A \ge \frac{3}{8}(2\pi A)$. Thus we can estimate $\tilde{\mathrm{S}}_i(v)$ bellow by $A(\frac{3/8-1/4}{3/8})$, so $Y_i(v) \ge \tau A$ provided $\tau < 1/3$. On the other hand, in the case $\mu \le 4$, $S_i(v) \ge \frac{5\pi A}{3}$, so $\frac{A^2}{\mu^2 S_i} \le \frac{3A}{5\pi}$. Because $R_i(v)$ consists of two components with the angular size of each of them at most $\frac{3}{2}\pi$, we get the lower estimate $\tilde{\lambda}(i,v) \ge \frac{(A-4\mu\pi)^2}{3/2\pi A} > \frac{2A}{3\pi} - 10\mu \ge \frac{5A}{8\pi}$ for large enough A. Thus again $Y_i(v) \ge \lambda(i,v) \ge \tau A$ provided $\tau < \frac{1}{20}$.

□

For a domain Ω and a ball B of radius $\frac{1}{\mu}$, let $N_{B,\Omega}(n,x)$ denote the maximal number of points v_j inside $B\cap\partial\Omega$ with $|v_j-v_k|\geq\exp(-\frac{A}{\mu^2}(n+1)-2\pi\gamma))$ if $j\neq k$, satisfying $\sum_{i=1}^n Y_i(v_j)\leq x\mu^2$ for all j, and $v+\exp\left(\cup_{j=1}^n R_j(v)\right)\cap B(v,\delta)=\emptyset$. Roughly speaking, $N_{B,\Omega}(n,x)$ is the maximal number of nonintersecting balls at certain distance to v which, by Lemma 3, are the candidates for having a large harmonic measure and rotation at least γ.

Let $N(n,x)=\sup N_{B,\Omega}(n,x)$, where the supremum is taken over all the domains Ω with $\operatorname{diam}\Omega\geq 10$ and all the balls B of the radius $\frac{1}{\mu}$.

Lemma 5.

$$N(n,x)\leq C_1(\gamma)e^{C_2x} \tag{19}$$

for some absolute constant C_2 and a constant $C_1(\gamma)$ depending only on γ.

Proof. The combinatorial construction we use here is very much similar to the one from [CM94].

We prove (19) by induction on n. The area counting shows that $N(n,x)\leq C(\gamma)\exp(2n\frac{A}{\mu^2})$. So, for small n (say, for $n=1,2$), the desired inequality is true for large enough $C_1(\gamma)$.

Let us now fix n. Assume that (19) is proved for $n-1$.

Fix a ball B of the unit radius and $N(n,x)$ points $v_j\in\mathcal{B}_1(v_0)$ satisfying the conditions of the Lemma.

Let $\psi=e^{-\sigma A}$ from Lemma 4 and $\eta=\min Y_1(v)$, where the minimum is taken over all $v\in\partial\Omega\cap B$.

Three cases are possible:

(1) $\eta\geq\psi$.
(2) $\eta\leq\psi^n$.
(3) $\psi^k\geq\eta\geq\psi^{k+1}$, for some $k=1,\dots,n-1$.

We show that in each of them $N(n,x)\leq C_1e^{C_2x}$, if C_1 and C_2 are large enough.

Case 1: $\boldsymbol{\eta\geq\psi}$. In this case for all v_j, we have

$$\sum_{i=2}^n Y_i(v_j)\leq\mu^2\left(x-\psi\right).$$

B can be covered by $\text{const}\times\exp\left(2\frac{A}{\mu^2}\right)$ balls of radius $\exp\left(-\frac{A}{\mu^2}\right)$. By rescaling by $\exp\left(\frac{A}{\mu^2}\right)$, we get that each of these balls contains no more then $N(n-1,x-\psi)$ points v_j.

So, by the induction hypothesis,

$$N_{B,\Omega}(n,x)\leq\text{const }N(n-1,x-\frac{\psi}{\mu^2})e^{2\frac{A}{\mu^2}}\leq\text{const }C_1e^{2\frac{A}{\mu^2}}e^{C_2(x-\frac{\psi}{\mu^2})}\leq C_1e^{C_2x}$$

provided $C_2>Ae^{\sigma A}$.

Case 2: $\boldsymbol{\eta\leq\psi^n}$. For some $v_0\in\partial\Omega\cap B$, we have $Y_1(v)\leq\psi^n$, so by Lemma 4, we have $Y_iv\geq A\tau$, $2\leq i\leq n$ for all $v\in v_0+\exp(R_1(v_0))$.

So, if $x<A\tau(n-1)$ then all the points v_j lie inside the ball $B(v,e^{-A})$. Thus, by rescaling by e^A, we have

$$N_{B,\Omega}(n,x)\leq N(n-1,x)\leq C_1e^{C_2x}.$$

If on the other hand $x\geq A\tau(n-1)$, then we can again use the area estimate to get

$$N(n,x)\leq\text{const }e^{2nA}\leq C_1e^{C_2x},$$

provided $C_2>\frac{10}{\tau}$ and C_1 is large enough.

Case 3: $\psi^k \geq \eta \geq \psi^{k+1}$, for some $k = 1, \ldots, n-1$. Again we take a point $v_0 \in B$ with $\mu^2\psi^{k+1} \leq Y_1(v_0) \leq \mu^2\psi^k$. As above, by Lemma 4, we have $Y_i(v) \geq A\tau\mu^3$, $2 \leq i \leq \frac{k}{\mu^2}$ for all $v \in v_0 + \exp(R_1(v_0))$.

Since $v_0 + \exp(R_1(v_0))$ is close to the logarithmic spiral with parameter γ, we can cover $v_0 + \exp(R_1(v_0) \cap \partial\Omega$ by $\operatorname{const} e^{kA}$ balls of the radius $\mu^2 e^{-kA}$. Then any point v_j either belongs to one of these balls or to $\mathcal{B}(v_0, e^{-A})$. This gives us, by the induction assumption,

$$\begin{aligned} N_{B,\Omega}(n,x) &\leq N(n-1, x-\psi^{k+1}) + \operatorname{const} e^{kA} N(n-k, x-(k-1)A\tau) \\ &\leq C_1 e^{C_2 x}\left(e^{-C_2\psi^{k+1}} + \operatorname{const} e^{kA - C_2(k-1)A\tau}\right) \leq C_1 e^{C_2 x} \end{aligned}$$

if $C_2 > 2e^{\sigma A}$ and $C_2\tau > 4$. □

Proof of Theorem 2. First, let us derive Theorem 2 from (19). We apply the estimate to a domain with $\operatorname{diam}\Omega = 10\mu$. Together with Lemma 3, (19) implies that

$$f(\alpha,\gamma) = \limsup_{n\to\infty} \frac{\log\left(N\left(n, \frac{C}{\sqrt{\gamma^2+1}} An\epsilon\right)\right)}{\sqrt{\gamma^2+1}(An+2\pi)} \leq C_0\epsilon$$

whenever $\exp(-\frac{A(n+1)}{\mu} - 2\pi\gamma)) < \delta \leq \exp(-\frac{An}{\mu} - 2\pi\gamma))$. Here C_0 is some absolute constant not depending on γ. Since

$$\alpha - \sqrt{(\alpha-1)^2+\gamma^2} = \frac{2\alpha - 1 - \gamma^2}{\alpha + \sqrt{(\alpha-1)^2+\gamma^2}} \geq 4\left(\alpha - \frac{1}{2} - \frac{\gamma^2}{2}\right) = \epsilon\,,$$

Theorem 2 follows from the estimate (19).

□

References

[Beu89] Arne Beurling, *The collected works of Arne Beurling. Vol. 1*, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 1989, Complex analysis, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer.

[BH08] Anton Baranov and Håkan Hedenmalm, *Boundary properties of Green functions in the plane*, Duke Math. J. **145** (2008), no. 1, 1–24.

[Bin] I. Binder, *Harmonic measure and rotation of planar domains*, Preprint.

[BP87] J. Becker and Ch. Pommerenke, *On the Hausdorff dimension of quasicircles*, Ann. Acad. Sci. Fenn. Ser. A I Math. **12** (1987), no. 2, 329–333.

[BS05] D. Beliaev and S. Smirnov, *Harmonic measure on fractal sets*, European Congress of Mathematics, Eur. Math. Soc., Zürich, 2005, pp. 41–59.

[CM94] Lennart Carleson and Nikolai G. Makarov, *Some results connected with Brennan's conjecture*, Ark. Mat. **32** (1994), no. 1, 33–62. MR MR1277919 (95g:30030)

[Fal90] Kenneth Falconer, *Fractal geometry*, John Wiley & Sons Ltd., Chichester, 1990, Mathematical foundations and applications.

[GM08] John B. Garnett and Donald E. Marshall, *Harmonic measure*, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2008, Reprint of the 2005 original.

[HS08] Håkan Hedenmalm and Alan Sola, *Spectral notions for conformal maps: a survey*, Comput. Methods Funct. Theory **8** (2008), no. 1-2, 447–474. MR MR2419488

[Jen70] James A. Jenkins, *On the Phragmén-Lindelöf theorem, the Denjoy conjecture, and related results*, Mathematical Essays Dedicated to A. J. Macintyre, Ohio Univ. Press, Athens, Ohio, 1970, pp. 183–200.

[JM95] P. W. Jones and Nikolai G. Makarov, *Density properties of harmonic measure*, Ann. of Math. (2) **142** (1995), no. 3, 427–455.

[Kra96] P. Kraetzer, *Experimental bounds for the universal integral means spectrum of conformal maps*, Complex Variables Theory Appl. **31** (1996), no. 4, 305–309.

[Mak85] N. G. Makarov, *On the distortion of boundary sets under conformal mappings*, Proc. London Math. Soc. (3) **51** (1985), no. 2, 369–384.

[Mak87] ———, *Conformal mapping and Hausdorff measures*, Ark. Mat. **25** (1987), no. 1, 41–89.

[Mak98] ———, *Fine structure of harmonic measure*, Algebra i Analiz **10** (1998), no. 2, 1–62.

[Pom92] Ch. Pommerenke, *Boundary behaviour of conformal maps.*, Fundamental Principles of Mathematical Sciences, 299, Springer-Verlag, Berlin, 1992.

[Zin00] Michel Zinsmeister, *Thermodynamic formalism and holomorphic dynamical systems*, SMF/AMS Texts and Monographs, vol. 2, American Mathematical Society, Providence, RI, 2000, Translated from the 1996 French original by C. Greg Anderson.

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