ASYMPTOTIC EXPANSION FOR THE INTEGRAL MIXED SPECTRUM OF THE BASIN OF ATTRACTION OF INFINITY FOR THE POLYNOMIALS $z(z+\delta)$

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Abstract. In this paper we establish asymptotic expansion for the integral mixed spectrum of the basin of attraction of infinity for the polynomials $z(z+\delta)$. It allows us to give another prove of the Ruelle's expansion for the dimension of the Julia set, as well to estimate the "prevalent" rate of rotation with respect to the Hausdorff measure of maximal dimension.

1. Introduction

The asymptotic expansion for the Hausdorff dimension of the Julia set of $z^2+\delta z$ was obtained by Ruelle in [10], who also proved that it depends analytically on δ. In his proof Ruelle used the dynamical ζ-function.

It was further improved by Collet, Dobbertin, and Moussa in [6] by the utilization of thermodynamic formalism. Some additional estimates on the asymptotic expansion were given by Baker and Stallard in [1]. They used the estimates of the spherical derivative of the polynomial $z^2+\delta z$.

Let us now recall the definition of the *Integral mixed spectrum*(see [3] for motivation and properties of the spectrum).

Let Ω be a simply connected domain, $z_0 \in \Omega$ and $\phi : \mathbb{D} \to \Omega$ be the Riemann map with $\phi(0) = z_0$, $\phi'(z_0) > 0$.

Definition 1. The *integral mixed spectrum* of the domain Ω is

$$m_\Omega(z) = \limsup_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^{z}(\zeta)|d|\zeta|}{\log \frac{1}{1-r}}.$$

In this paper we obtain the asymptotic expansion for the integral mixed spectrum of quadratic Julia sets.

Theorem 1.

$$m_{A_\delta(\infty)}(z) = \frac{|z|^2|\delta|^2}{16\log 2} + \frac{\Re(\overline{z}\delta)|\delta|^2|z|^2}{64\log 2} - \frac{|z|^2||\delta|^2\cos\arg(\delta)}{32\log 2} + \boldsymbol{O}(|z|^2|\delta|^4),$$

where $A_\delta(\infty)$ is the basin of attraction of infinity for the polynomial $f_\delta(z) = z(z+\delta)$.

Combined with the Bowen's formula it allows us to estimate the Hausdorff dimension of the Julia sets, re obtaining the asymptotic from [6].

Corollary 1.

$$d = \dim \partial A_\delta(\infty) = 1 + \frac{|\delta|^2}{16\log 2} - \frac{|\delta|^3\cos\arg\delta}{64\log 2} + \mathbf{O}(|\delta|^4)$$

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Theorem 1 also has an interesting geometric consequence. By [3], we can rewrite the dimension spectrum (see the definition below) in terms of the integral means spectrum. Then the asymptotic of the previous theorem allows us to describe the "prevalent" rotation on the Julia set.

Theorem 2. *Let κ denote the Hausdorff measure of the full dimension on the Julia set. Then at κ-a.e. the Julia set behaves like a logarithmic spiral with parameter*

$$\gamma = \frac{|\delta|^3}{64\log 2}\sin\arg\delta + \boldsymbol{O}(|\delta|^4),$$

in the sense that the radial behavior of the Riemann map is

$$\arg {\phi'}_\delta(r\zeta) \sim \gamma \log\frac{1}{1-r}.$$

2. Proofs

Let us start with the proof of the corollary 1. We need to start with a few definitions and statements from [3].

Let $x \in \partial\Omega$, $\delta > 0$, Ω_δ be the connected component of the set

$$\{y \in \Omega : |x-y| > \delta\}$$

containing x_0.

Define the *rotation of the domain Ω near x at distance δ* as

$$\rho(x,\delta) = \exp\left(\inf_{y\in\partial\Omega_\delta, |y-x|=\delta} \arg(y-x)\right),$$

where the branch of the arg is selected so that $-\pi < \arg(x_0 - x) \le \pi$.

Definition 2. Let Ω be a domain with the boundary invariant under hyperbolic dynamics. Then the *dimension mixed spectrum* of Ω is defined as

$$f(\alpha,\gamma) = \dim\left\{x\in\partial\Omega : \lim_{\delta\to 0}\frac{\log\omega(B(x,\delta))}{\log\delta} = \alpha;\ : \lim_{\delta\to 0}\frac{\log rho(x,\delta)}{\log\delta} = \gamma\right\}.$$

Here $B(x,\delta)$ is a disk of radius δ around x, dim is the Hausdorff dimension.

Theorem 3 ([3])**.** *Let Ω be a Jordan repeller. Then*

1. *Integral and dimension mixed spectra are related by a Legendre type transform:*
$$f_\Omega(\alpha,\gamma) = \inf_z\left(\alpha m_\Omega(z) + (1-\alpha)\Re(z) - \gamma\Im(z) + \alpha\right)$$
$$m_\Omega(z) = \sup_{\alpha,\gamma}\left(\frac{f_\Omega(\alpha,\gamma) - (1-\alpha)\Re(z) + \gamma\Im(z) - \alpha}{\alpha}\right).$$
2. *$f_\Omega(\alpha,\gamma)$ and $m_\Omega(z)$ are real-analytic.*
3. *Either the boundary $\partial\Omega$ is real analytic and all the mixed spectra are trivial, or $m_\Omega(z)$ is a strictly convex function, $f_\Omega(\alpha,\gamma)$ is strictly concave on the domains where it is not equal to $-\infty$.*

   In particular, the rotational part of the integral mixed spectrum is nontrivial iff the real part is nontrivial.

Lemma 1. *Let Ω be the basin of attraction of infinity of a hyperbolic polynomial of degree d. Then*

$$m_\Omega(z) = -\frac{1}{\log d}P\left(\Re\left(\bar{z}\frac{\log F'\circ\phi}{\log B'}\right)\right) - 1.$$

Proof of the corollary 1. $\partial A_\delta(\infty)$ is a quasi circle. So, by Theorem 3.1 from [8] (see also [9]),

$$m_{A_\delta(\infty)}(d) = d - 1.$$

Using the asymptotic from Theorem 1, it can be rewritten as

$$\frac{d^2|\delta|^2}{16\log 2} + \frac{d^3|\delta|^3}{64\log 2} - \frac{d^2|\delta|^2\cos\arg(\delta)}{32\log 2} + \mathbf{O}(d^2|\delta|^4) = d - 1.$$

The statement of the corollary is just the asymptotic expansion for the solution of the last equation. □

Let us now prove Theorem 2.

Proof of Theorem 2. As established in [3], the dimension spectrum is a strictly concave function (because $m_{A_\delta(\infty)}(z) \not\equiv 0$, so $\sqcup^2 \neq 0$). So there is a unique point α_0, γ_0 with $f(\alpha_0, \gamma_0) = \dim A_\delta(\infty) =: d$. So, by the definition of the dimension spectrum, for any ϵ

$$\dim\Big\{ z \in \partial A_\delta(\infty) : \text{there exists a sequence } r_n \to 0 \text{ such that} \quad \lim_{n\to\infty} \frac{\rho(z, r_n)}{-\log r_n} - \gamma_0| > \epsilon \Big\} < d.$$

So to prove the theorem, we need to establish the asymptotic for γ_0.

But

$$f(\alpha_0, \gamma_0) = \alpha_0 m(d) - \alpha_0 d + \alpha + d.$$

So, the function $l(d_0, d_1) = \alpha_0 m(d_0 + id_1) - \alpha_0 d_0 - \gamma_0 d_1 + d_0 + \alpha_0$ reaches its minimum at $d_0 = d,\ d_1 = 0$. It means that $\frac{\partial l}{\partial d_0}(d, 0) = \frac{\partial l}{\partial d_1}(d, 0) = 0$. Thus

$$\gamma_0 = \frac{\frac{-i\partial m}{\partial d_1}(d, 0)}{1 - \frac{\partial m}{\partial d_0}(d, 0)}.$$

By Corollary 1, $d = 1 + \mathbf{O}(|\delta|^2)$. Hence, by the expansion in Theorem 1,

$$\frac{-i\partial m}{\partial d_1}(d, 0) = \frac{|\delta|^2 d^2 \sin\arg\delta}{64\log 2} + \mathbf{O}(d|\delta|^4) = |\delta|^2 \sin\arg\delta \frac{1}{64\log 2} + \mathbf{O}(|\delta|^4)$$

and

$$\frac{\partial m}{\partial d_0} = \mathbf{O}(|\delta|^2).$$

Now the theorem follows from the last two equations. □

We can now turn to the proof of the 1.

To get the asymptotic for the $m_{A_\delta(\infty)}(z)$, let us first get the asymptotic for the Riemann map of the $A_\delta(\infty)$.

Let ϕ_δ be the Riemann map $\phi_\delta : \mathbb{D}_- \to A_\delta(\infty)$ conjugating $z^2 + \delta z$ and z^2:

$$\phi_\delta^2(z) + \delta\phi(z) = \phi_\delta(z^2), \quad \phi'(\infty) = 1. \tag{1}$$

For the existence of such a map, see, for example, [5], Chapter VIII. Since the map $\phi_\delta(z)$ satisfies the functional equation (1), it is analytic in both z and δ. By the theorems of Mañé, Sad, and Sullivan (see [7]),Sullivan and Thurston (see [11]) and Bers and Royden (see [2]), the family ϕ_δ can be extended to an analytic family of quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$.

So we can write

$$\phi_\delta(z) = z + \delta g_1(z) + \delta^2 g_2(z) + \delta^3 g_3(z, \delta),$$

where the maps g_1, g_2 are independent on δ, and $g_3(z,\delta)$ is bounded independent on δ in the α-Hölder metric for any $\alpha < 1$.

Let us draw a few consequences of (1).

Lemma 2.

$$g_1(z^2) = z + 2zg_1(z) \tag{2}$$

$$g_2(z^2) = g_1(z) + 2zg_2(z) + g_1(z)^2. \tag{3}$$

Proof. By the equation (1),

$$\begin{aligned}\phi_\delta(z^2) = z^2 + \delta g_1(z^2) + \delta^2 g_2(z^2) + \mathbf{O}(\delta^3) &= \phi_\delta(z)(\phi_\delta(z)+\delta)\\ = \left(z + \delta g_1(z) + \delta^2 g_2(z) + \mathbf{O}(\delta^3)\right)&\left(z + \delta(1+g_1(z)) + \delta^2 g_2(z) + \mathbf{O}(\delta^3)\right)\\ = z^2 + \delta(g_1(z) + z(1+g_1(z))) &+ \delta^2(2zg_2(z) + g_1(z) + g_1^2(z)) + \mathbf{O}(\delta^3)\end{aligned}$$

Now all the statements of the lemma may be obtained by comparing the coefficients of δ^k $(k=1,2)$. □

Let $W(z) = \sum_{k=0}^{\infty} 2^{-k} z^{2^k}$ be the Weierstrass function.

Lemma 3.

$$g_1(z) = -\frac{z}{2}\, W(\overline{z}). \tag{4}$$

Proof. Let $g_1(z) = \sum_{k=0}^{\infty} c_k z^{-k}$. Then, by 2,

$$\sum_{k=0}^{\infty} c_k z^{-2k} = z + \sum_{k=0}^{\infty} 2c_k z^{1-k}.$$

Hence we have

$$\begin{aligned} c_0 &= -\frac{1}{2}\\ c_{2k} &= 0, \quad k \geq 1\\ c_{2k+1} &= \frac{1}{2}c_k \quad k \geq 0.\end{aligned}$$

So, by induction we get

$$c_{2^l-1} = -2^{-l-1} \quad l \geq 0$$

and all other coefficients are equal to 0. □

We say that two α-Hölder functions $f_1(z)$, $f_2(z)$ on $\mathbb{T}$ are equivalent, $f_1 \sim f_2$ if $f_1(z) = f_2(z) + u(z^2) - u(z)$ for some α-Hölder function u. The main use of this equivalence is the fact that if $f_1 \sim f_2$, then $P(f_1) = P(f_2)$. Here the pressure is taken with respect to $z \mapsto z^2$.

Let us derive a few equivalences for the coefficients in the expansion of ϕ_δ.

Lemma 4.

$$\frac{g_1(z)}{z} \sim -\frac{1}{z} \tag{5}$$

$$\frac{g_1(z)^2}{z^2} \sim -\frac{1}{3}\left(\frac{4g_1(z)}{z^2} + \frac{1}{z^2}\right) \tag{6}$$

$$\frac{g_2(z)}{z} \sim \frac{1}{3}\left(\frac{g_1(z)}{z^2} + \frac{1}{z^2}\right). \tag{7}$$

Proof. The relation (5) follows from (2), which can be rewritten as

$$\frac{g_1(z^2)}{z^2} - \frac{g_1(z)}{z} = \frac{1}{z} + \frac{g_1(z)}{z}.$$

To get (6) we write (2) in the form

$$\left(\frac{g_1(z^2)}{z^2}\right)^2 - \left(\frac{g_1(z)}{z}\right)^2 = \frac{3g_1^2(z)}{z^2} + \frac{1}{z^2} + \frac{4g_1(z)}{z^2}.$$

If we write (3) in the form

$$\frac{g_2(z^2)}{z^2} - \frac{g_2(z)}{z} = \frac{g_1(z)}{z^2} + \frac{g_2(z)}{z} + \frac{g_1^2(z)}{z^2},$$

we get, using (6),

$$\frac{g_2(z)}{z} \sim -\frac{g_1(z)}{z^2} - \frac{g_1^2(z)}{z^2} \sim -\frac{g_1(z)}{z^2} + \frac{1}{3z^2} + \frac{4}{3}\frac{g_1(z)}{z^2}$$

which gives us (7). □

To compute the asymptotic of $m_{A_\delta(\infty)}(z)$, we will need the asymptotic of $\psi_\delta(z) = \log\left(\bar{z}\left(\phi_\delta(z) + \frac{\delta}{2}\right)\right)$.

Lemma 5.

(8) $$\psi_\delta(z) = \psi_0(z) + \mathbf{O}(|\delta|^3),$$

where

$$\psi_0(z) \sim -\frac{1}{2}\left(\delta\bar{z} + \delta^2\left(\frac{1}{2}\bar{z}W(\bar{z})\right) - \frac{3}{4}\bar{z}\right).$$

Proof.

$$\psi_\delta(z) = \log\left(1 + \delta\bar{z}\left(g_1(z) + \frac{1}{2}\right) + \delta^2\bar{z}g_2(z) + \mathbf{O}(|\delta^3|)\right)$$
$$= \delta\bar{z}\left(g_1(z) + \frac{1}{2}\right) + \delta^2\bar{z}g_2(z) - \frac{\delta^2\bar{z}^2}{2}\left(g_1(z) + \frac{1}{2}\right)^2 + \mathbf{O}(|\delta|^3).$$

By (5),

$$\delta\bar{z}\left(g_1(z) + \frac{1}{2}\right) \sim \frac{\bar{z}}{2} - \bar{z} = -\frac{\bar{z}}{2}.$$

By equations (6) and (7)

$$\delta^2\left(\bar{z}g_2(z) - \frac{1}{2}\delta^2\bar{z}^2\left(g_1(z) + \frac{1}{2}\right)^2\right) \sim \delta^2\left(\frac{3}{8}\bar{z}^2 + \frac{1}{2}\bar{z}^2 g_1(z)\right) = \delta^2\left(\frac{3}{8}\bar{z} - \frac{\bar{z}}{4}W(\bar{z})\right).$$

This proves the lemma. □

Now we can start the estimation of $m_{A_\delta(\infty)}(z)$.
By Lemma 1,

(9) $$m_{A_\delta(\infty)}(w) = \frac{1}{\log 2}P\left(\Re\left(\bar{w}\log\frac{F'_\delta \circ \phi_\delta}{2z}\right)\right) - 1$$
$$= \frac{1}{\log 2}P\left[\Re\left(\bar{w}\log\bar{z}\left(\phi_\delta(z) + \frac{\delta}{2}\right)\right)\right] - 1 = \frac{1}{\log 2}P\left(\bar{w}\psi_\delta(z)\right).$$

Let us note that if $f_1(z) \equiv f_2(z)$, then $\Re(\bar{w}f_1(z)) \equiv \Re(\bar{w}f_2(z))$. So, by Lemma 5

$$(10)\quad P(\bar{w}\psi_\delta(z)) = P(\Re(\bar{w}\psi_0(z)) + \mathbf{O}(|\delta|^3))$$
$$= P\left(-\frac{1}{2}\Re\left(\delta\overline{wz} + \delta^2\left(\frac{1}{2}\overline{zw}W(\bar{z}) - \frac{3}{4}\overline{zw}\right)\right) + \mathbf{O}(|w||\delta|^3)\right).$$

Let $\alpha = \arg\delta,\ \beta = \arg w$, and denote

$$\psi_1(z) = -\frac{1}{2}\Re(e^{i(\alpha-\beta)}\overline{z})$$
$$\psi_2(z) = \frac{1}{2}\Re\left(e^{i(2\alpha-\beta)}\frac{1}{2}\bar{z}W(\bar{z}) - \frac{3}{4}\bar{z}\right)$$
$$\psi_3(z) = \frac{\Re(\bar{w}(\psi_\delta(z) - \psi_0(z)))}{|\delta|^3}.$$

Then,

$$(11)\qquad P(\bar{w}\psi_\delta(z)) = P(|w|(|\delta|\psi_1 + |\delta|^2\psi_2 + |\delta|^3\psi_3)).$$

Now we use the fact that

$$(12)\qquad P(t\Gamma) = \log 2 + \lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} e^{tS_n\Gamma(z)},$$

where $S_n(\Gamma)(z) = \sum_{k=0}^n \Gamma(z^{2^k})$. It follows from the first description of the pressure in [4]. Thus, since the convergence in (12) is uniform, we can write

$$(13)\quad P(t\Gamma) = \log 2 + t\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n\Gamma + \frac{t^2}{2}\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}(S_n\Gamma)^2$$
$$+\frac{t^3}{6}\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}(S_n\Gamma)^3 + \mathbf{O}\left(t^4\sup_z|S_n\Gamma(z)|^4\right).$$

Applying (13) to (11) we get

$$(14)\quad P(\bar{w}\psi_\delta(z)) = \log 2 + |z|\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n\psi_\delta + \frac{|z|^2}{2}\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}(S_n\psi_\delta)^2$$
$$+\frac{|z|^3}{6}\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}}(S_n\psi_\delta)^3 + \mathbf{O}\left(|z|^4\sup_z|S_n\psi_\delta(z)|^4\right).$$

Let us note that

$$\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n\psi_\delta = \log 2m'(0) = 0.$$

So we can rewrite (14) as

$$(15)\quad P(\bar{w}\psi_\delta(z)) =$$
$$\log 2 + \frac{1}{2}\left[|z|^2|\delta|^2\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n(\psi_1)^2 + |z|^2|\delta|^3\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n(\psi_1)(\psi_2)\right]$$
$$-\frac{1}{6}|z|^3|\delta|^3\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n(\psi_1)^3 + \mathbf{O}(|z|^2|\delta|^4).$$

The desired asymptotic for the m follows now from the next lemma.

Lemma 6.

$$\int_{\mathbb{T}} (S_n\psi_1)^2 = \frac{n}{4} \tag{16}$$

$$\lim_{n\to\infty} \frac{1}{n}\int_{\mathbb{T}} (S_n\psi_1)(S_n\psi_2) = -\frac{1}{32}\cos\alpha \tag{17}$$

$$\int_{\mathbb{T}} (S_n\psi_1)^3 = \frac{3}{32}(n-1)\cos(\alpha-\beta). \tag{18}$$

Proof. Let us rewrite ψ_1 and ψ_2

$$\psi_1(e^{i\theta}) = \frac{1}{2}\cos(\theta-\alpha+\beta)$$

$$\psi_2(e^{i\theta}) = \frac{1}{2}\sum_{k=0}^{\infty} c_k \cos((2^k+1)\theta+\beta-2\alpha) \quad c_k = \begin{cases} \frac{1}{8}, & k=0 \\ 2^{-k-2}, & k>0 \end{cases}.$$

Then

$$S_n\psi_1(e^i\theta) = \sum_{k=0}^{n} \cos(2^k\theta+\beta-\alpha).$$

$\int \cos(a\theta+b)\cos(c\theta+d) \neq 0$ if and only if $a = \pm c$ (for integer $a,\ c$).
This implies that

$$\int_{\mathbb{T}} S_n(\psi_1)^2 = \frac{1}{4}\sum_{k=0}^{n}\int_{\mathbb{T}} \cos^2(2^k\theta+\beta-\alpha) = \frac{n}{4}$$

and

$$\int_{\mathbb{T}} S_n(\psi_1)^3 = \frac{1}{8}\sum_{k_1,k_2,k_3=0}^{n}\int_{\mathbb{T}} \cos(2^{k_1}\theta+\beta-\alpha)\cos(2^{k_2}\theta+\beta-\alpha)\cos(2^{k_3}\theta+\beta-\alpha).$$

Since $\cos x\cos y = \frac{1}{2}(\cos(x+y)+\cos(x-y))$, the integral $\int_{\mathbb{T}} \cos(2^{k_1}\theta+\beta-\alpha)\cos(2^{k_2}\theta+\beta-\alpha)\cos(2^{k_3}\theta+\beta-\alpha)$ is not equal to 0 if and only if $2^{k_1} = 2^{k_2}+2^{k_3}$, or $2^{k_2} = 2^{k_1}+2^{k_3}$, or $2^{k_3} = 2^{k_1}+2^{k_2}$. So

$$\int_{\mathbb{T}} S_n(\psi_1)^3 = \frac{3}{8}\sum_{k=0}^{n-1}\int_{\mathbb{T}} \cos(2^{k+1}\theta+\beta-\alpha)\cos(2^k\theta+2\beta-2\alpha) = \frac{3}{32}(n-1)\cos(\beta-\alpha).$$

To prove (17), let us recall that by the ergodic theorem,

$$\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n(\psi_1)(\psi_2) = \int_{\mathbb{T}}\psi_1\psi_2 + \sum_{k=0}^{\infty}\int_{\mathbb{T}}\psi_1(z)\psi_2(z^{2^k}) + \sum_{k=0}^{\infty}\int_{\mathbb{T}}\psi_2(z)\psi_1(z^{2^k}).$$

Let us note that all the coefficients in front of θ in the expansions of ψ_1 and ψ_2 are odd, so all the integrals $\int_{\mathbb{T}}\psi_2(z)\psi_1(z^{2^k})$ and $\int_{\mathbb{T}}\psi_1(z)\psi_2(z^{2^k})$ are equal to 0. It means that

$$\lim_{n\to\infty}\frac{1}{n}\int_{\mathbb{T}} S_n(\psi_1)(\psi_2) = \int_{\mathbb{T}}\psi_1\psi_2 = -\frac{1}{16}\int_{\mathbb{T}}\cos(\theta+\beta-\alpha)\cos(\theta+\beta-2\alpha) = -\frac{1}{32}\cos\alpha.$$

□

Now we plug the expansion from the last lemma in equation (15), and then in equation (9) to get the asymptotic expansion in Theorem 1.

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