HARMONIC MEASURE AND ROTATION OF SIMPLY CONNECTED PLANAR DOMAINS

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Abstract. In this paper we introduce the notion of *Mixed spectrum of harmonic measure*. The spectrum captures both the local dimension of harmonic measure and the rate of rotation of Green lines near the boundary. We investigate the main properties of different versions of the spectrum and prove the interrelation between them. In addition, we establish the real-analyticity of the spectra for domains with boundary invariant under a hyperbolic dynamics. We show that the sharp bounds on the mixed spectrum for general bounded planar domains are the same as for the dynamically-invariant domains. It allows us to establish that the universal bounds for the different mixed spectra of simply-connected domains are related by a Legendre-type transform. We also describe all possible spectra of simply-connected domains, and relate sharp bounds on the mixed spectra for different classes of planar domains to the ones for bounded simply-connected domains.

1. Introduction

Let Ω be a simply connected domain, $z_0 \in \Omega$, and $\phi : \mathbb{D} \to \Omega$ be the Riemann map with $\phi(0) = z_0$, $\phi'(z_0) > 0$. In this paper we study fine properties of the rotation of the images of the radii under the map ϕ and investigate the relation with the properties of harmonic measure.

For a set $K \subset \partial\Omega$ the harmonic measure $\omega(k, z_0, \Omega)$ of K at z_0 is given by $m(\phi^{*-1}(K))$, where m is the normalized Lebesgue measure on the unit circle $\mathbb{T}$, ϕ^* is the radial boundary values of ϕ. In the rest of the paper we fix some $z_0 \in \Omega$. The results will not depend on the choice of z_0.

We will be interested mainly in the *rate of decay*, or *local dimension* of harmonic measure near a given point $z \in \partial\Omega$. Roughly speaking, it is a power $\alpha(z)$ such that the harmonic measure of the disc $B(z, r)$ of radius r behaves like $r^{\alpha(z)}$ when $r \to 0$. It is easy to see that $\alpha(z)$ does not depend on the choice of z_0. Note that for most of the points $\alpha(z)$ is not well defined.

The "prevalent" rate of decay is closely related to the question of the dimension of harmonic measure. Namely the fact that for a simply connected domain ω-almost everywhere the rate of decay is well defined and equal to one is equivalent to the Makarov's theorem about the dimension of harmonic measure ([10]). The latter asserts that the dimension of harmonic measure is equal to 1. Moreover, the theorem provides the precise gauge function for the dimension of harmonic measure. It states that it is always absolutely continuous with respect to the Hausdorff measure with the weight

$$h(t) = t \exp\left(C \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}} \right), \tag{1}$$

where C is an absolute constant.

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It is known that $\alpha(z)$ is closely related to the rate of growth of the derivative of the conformal map $\phi : \mathbb{D} \to \Omega$ (see [12]). Naively one would expect, for a suitable $\zeta \in \mathbb{D}$, that

$$\left|\phi'(\zeta)\right| \approx \frac{2r}{\omega(B(z,r))}, \text{ and } 1 - |\zeta| \approx \omega(B(z,r)),$$

and this can be made precise (see [12]).

Note that this observation allows one to deduce Makarov's dimension theorem from an estimate on the growth of the Bloch function applied to the $\log|\phi'(\zeta)|$.

$\alpha(z)$ has a geometric meaning. If the domain Ω looks like an angle of the opening β near z, then $\alpha(z) = \frac{\pi}{\beta}$. Thus the quantity $\frac{\pi}{\alpha(z)}$ can be regarded as the generalized angle of Ω near z.

Now let us introduce our second object – *rotation.* Let $z \in \partial\Omega$, $\delta > 0$, Ω_δ be the connected component of the set $\{y \in \Omega : |z - y| > \delta\}$ containing z_0. Define the *rotation of the domain* Ω *near* z *at distance* δ as

$$\rho(z, \delta) = \exp\left(\inf_{y \in \partial\Omega_\delta, |y-z|=\delta} \arg(y - z)\right),$$

where the branch of the arg is selected so that $-\pi < \arg(z_0 - z) \leq \pi$.

We need to take an exponent here so that the rotation will scale like the harmonic measure. Roughly speaking, the rotation measures how many times a curve must rotate around z to reach the r-neighborhood of z from z_0.

We will be interested in the *rotation speed around* z, $\gamma(z)$. Roughly speaking, it is such a number that $\rho(x, \delta)$ behaves like $\delta^{\gamma(z)}$. $\gamma(z)$ can be both positive and negative, depending on the direction of the rotation. In general, $\gamma(z)$ is again not well defined.

Note that one can measure the rotation along the Green line by computing $\arg\phi'$. Naively, one would expect that for some point $\zeta \in \mathbb{D}$

$$\left|\phi'(\zeta)^{-i}\right| \approx \rho(z, r), \text{ and } 1 - |\zeta| \approx \omega(B(z, r)).$$

Again one can make this estimate precise in certain situations, for example, for quasi circles with the small quasiconformal constant.

One can also use this observation to prove that the rotation is trivial almost everywhere by harmonic measure. To obtain this statement one just needs to apply Makarov's estimate on the growth of the Bloch function to the function $\arg\phi'(\zeta)$.

Sharp estimates on the growth of $\arg\phi'$ are given by Goluzin's rotation theorem, see [5]. It is also worth mentioning McMillan's non twisting theorem, which states that for almost all radii either the image of the radius is very twisted around the endpoint or the Riemann map is conformal at this point (see [19]).

To give a quantitative metric description of the rotation phenomenon, we introduce the notion of the *dimension mixed spectrum.* The latter is a continuum of parameters $f(\alpha, \gamma)$, $\alpha > 0, \gamma \in \mathbb{R}$, characterizing "harmonic" dimension (i.e., dimension with respect to harmonic measure) of the boundary set with a prescribed speed of rotation of Green lines and prescribed decay of the harmonic measure. Roughly, $f(\alpha, \gamma)$ is the dimension of the set

$$S_{\alpha,\gamma} = \{z \in \partial\Omega : \alpha(z) = \alpha,\ \gamma(z) = \gamma\}.$$

The precise definition has different versions, involving lim sup or lim inf, and Hausdorff or Minkowski dimensions (see Section 2).

Dimension mixed spectra provide a multitude of information about the geometry of the $\partial\Omega$. One can extract from them information about all possible rates of decay and rotation speeds, prevalent rotation speed and decay of harmonic measure in the sense of Hausdorff measures of different dimensions.

The motivation for the introduction of the dimension mixed spectrum comes from the multifractal analysis. The latter was introduced in the middle eighties in a few physical papers, for example, in

[8]. The idea of the method is that to study properties of the measure, we consider its dimension spectrum, that is the continuum of parameters characterizing the size of the set where a certain power law applies to mass concentration. An overview of the method from both mathematical and physical viewpoints can be found in [7]. The multifractal analysis of the planar harmonic measure was carried out by Makarov in [13].

Another way to view the mixed spectra is to consider them as a generalization of the pressure defined in conformal dynamics with the use of machinery of thermodynamic formalism (see [25]). We discuss this in more detail in section 3.

Yet another motivation for introduction of multifractal spectra comes from the dimension theory (see [16]).

In Section 2 we also introduce *integral mixed spectrum* (Definition 2). For the discussion of the basic properties of the integral mixed spectrum see Section 2.

We also investigate its connections to the "dual" (in the sense of large deviations) object, the *Minkowski distortion mixed spectrum* $d(a, b)$. Roughly speaking, the latter is a two dimensional set of parameters measuring the Minkowski dimension of the set where $\log \phi'(r\zeta)$ has prescribed growth (see Definition 4). As usual in the theory of large deviations, it is related to the integral mixed spectrum by a Legendre type transform (Lemma 3). If instead of the Minkowski dimension we consider the Hausdorff one, we obtain the *Hausdorff distortion mixed spectrum* $\tilde{d}(a, b)$ (Definition 5). It is always no greater then Minkowski distortion spectrum (Lemma 4).

We also obtain the relation between dimension and distortion spectra. Namely, we obtain the following theorem.

Theorem 1. *For any simply connected domain* Ω *,*

$$d_\Omega(a, b) \geq (1 - a) f_\Omega \left(\frac{1}{1 - a}, \frac{-b}{1 - a} \right).$$

It is interesting to compare the theorem with the results of [1].

It was noticed by Carleson in [3] that the harmonic measure on the repellers of certain dynamical systems is closely related to the underlying dynamics. In [15] the harmonic measure for some Fatou components was studied. Further study of the harmonic measure of the domains with invariant boundary was done in [22], [21] and [23], where the harmonic measure was studied using the dynamics in the neighborhood of unit circle induced by the dynamics of the boundary. On the other hand, Carleson's original approach was extended by Makarov in [11] and [12].

All of the spectra defined in the previous section behave nicely for simply connected domains with the boundaries (or "essential" parts of them) invariant under a hyperbolic dynamic. We call such domains *Jordan repellers*. One example of such a domain is the basin of attraction to infinity of a hyperbolic polynomial. Another example is a snowflake domain, or Carleson fractal. It is constructed in the following way.

Let P be a polygon (not necessarily convex), and let an interval I contain one of the sides q of the polygon and does not have any other intersections with P. For each side p_i of the polygon not intersecting I, consider the linear mapping ϕ_i with $\phi_i(I) = p_i$ and the $\phi_i(P)$ is facing outside of the P.

Now let $P_0 = P \cap I - q$, and

$$P_{n+1} = P_n \cap_{i_1,\ldots,i_n} \phi_{i_1} \ldots \phi_{i_n}(P) - \cap_{i_1,\ldots,i_n} \phi_{i_1} \ldots \phi_{i_n}(q).$$

The P_n is a Jordan curve. Assume that the limit (in the Caratheodory sense, see [19]) of the curves P_n is also a Jordan curve J. Let now Γ be an analytic curve joining the ends of the I. Assume that Γ does not intersect the inside of J, and that $\Gamma \cap I$ is a C^∞ curve. Then we consider the domain Ω bounded by $J \cap \Gamma$.

This example is of special importance to us. The reason for this is that any domain can be approximated, in the sense of mixed spectra, by snowflakes (Theorem 3).

The mixed spectra for Jordan repellers are thermodynamic objects. They all can be defined in terms of the pressures of some potentials related to the dynamics on the boundaries and the dimensions of the corresponding Gibbs measures. This fact allows us to use the extensive techniques of thermodynamic formalism. For example one apply Ruelle-Perron-Frobenius transfer operator, Ruelle's ζ-function and Patterson-Sullivan conformal measures to fully understand the relations between the mixed spectra and their possible behavior. The results obtained by the author in this direction are summarized in the following theorem.

Theorem 2. *Let Ω be a Jordan repeller. Then*

(1) *Both Hausdorff and Minkowski versions of the dimension mixed spectra for α, γ are equal to*

$$\dim\left\{z \in \partial\Omega \ : \ \lim_{\delta\to 0}\frac{\omega(B(z,\delta))}{\log\delta} = \alpha; \ \lim_{\delta\to 0}\frac{\rho(z,\delta)}{\log\delta} = \gamma\right\}.$$

(2) *The integral mixed spectrum exists as a limit:*

$$m_\Omega(z) = \lim_{r\to 1-}\frac{\int_{r\mathbb{T}}|\phi'(\zeta)^z|d\zeta}{\log\frac{1}{1-r}}.$$

(3) *Integral and dimension mixed spectra are related by a Legendre type transform:*

$$f_\Omega(\alpha,\gamma) = \inf_z\left(\alpha m_\Omega(z) + (1-\alpha)\Re(z) - \gamma\Im(z) + \alpha\right)$$
$$m_\Omega(z) = \sup_{\alpha,\gamma}\left(\frac{f_\Omega(\alpha,\gamma) - (1-\alpha)\Re(z) + \gamma\Im(z) - \alpha}{\alpha}\right).$$

(4) *Both Hausdorff and Minkowski versions of the distortion mixed spectra for a, b are equal to* $\dim\left\{\zeta \in \partial\mathbb{D} \ : \ \lim_{r\to 1}\frac{\log\phi'(r\zeta)}{\log(1-r)} = a + ib\right\}$.

(5) *$f_\Omega(\alpha,\gamma)$, $d_\Omega(a,b)$, and $m_\Omega(z)$ are real-analytic.*

(6) *Either the boundary $\partial\Omega$ is real analytic and all the mixed spectra are trivial, or $m_\Omega(z)$ is a strictly convex function, $f_\Omega(\alpha,\gamma)$ and $d_\Omega(a,b)$ are strictly concave on the domains where they are not equal to $-\infty$.*

In particular, the rotational part of the integral mixed spectrum is nontrivial iff the real part is nontrivial.

The last statement of the theorem implies, in particular, that, unless the boundary of Jordan repeller is an analytic curve, it has nontrivial rotation. On a large scale it can be seen on computer pictures, but on a small scale it goes undetected by a computer. The theorem is proved in Section 3.

In Section 4 we investigate the sharp bounds for mixed spectra or *universal spectra*.

Definition 1. Let Υ be a set of simply connected domains (or, equivalently, their Riemann maps). We define the *Universal Integral Mixed Spectrum* for Υ by

$$\mathbf{M}_{(\Upsilon)}(z) = \sup_{\Omega\in\Upsilon} m_\Omega(z).$$

By the same token we can define universal mixed Hausdorff dimension ($\tilde{\mathbf{F}}_{(\Upsilon)}(\alpha,\gamma)$), universal mixed Minkowski dimension ($\mathbf{F}_{(\Upsilon)}(\alpha,\gamma)$), universal mixed Hausdorff distortion ($\tilde{\mathbf{D}}_{(\Upsilon)}(a,b)$), and universal mixed Minkowski distortion ($\mathbf{D}_{(\Upsilon)}(a,b)$) spectra for Υ.

The main tool in the chapter is *fractal approximation*.

The first result of this type was obtained by Carleson and Jones in [4]. They proved, in our notation, that $\mathbf{M}_{(\mathrm{Bnd})}(1) = \mathbf{M}_{(\mathrm{Snowflakes})}(1)$. In [13] Makarov obtained this equality for an arbitrary real t. We prove the following theorem

Theorem 3.

$$\boldsymbol{M}_{(Bnd)}(z) = \boldsymbol{M}_{(Snowflakes)}(z),$$

where (Bnd) is the set of all bounded simply connected domains, $(Snowflakes)$ is the set of all snowflakes.

The theorem combined with the results of Section 2 shows that universal mixed spectra have some of the properties of the spectra of a snowflake.

First of all, it provides the fractal approximation for other spectra.

Corollary 1.

$$\tilde{\boldsymbol{D}}_{(Bnd)}(a,b) = \boldsymbol{D}_{(Bnd)}(a,b) = \boldsymbol{D}_{(Snowflakes)}(a,b),$$
$$\boldsymbol{F}_{(Bnd)}(\alpha,\gamma) = \tilde{\boldsymbol{F}}_{(Bnd)}(\alpha,\gamma) = \boldsymbol{F}_{(Snowflakes)}(\alpha,\gamma).$$

It also shows that relations between the universal spectra are the same as in the case of a snowflake.

Corollary 2.

$$\boldsymbol{F}_{(Bnd)}(\alpha,\gamma) = \inf_z \left(\alpha \boldsymbol{M}_{(Bnd)}(z) + (1-\alpha)\Re(z) - \gamma\Im(z) + \alpha\right)$$
$$\boldsymbol{M}_{(Bnd)}(z) = \sup_{\alpha,\gamma}\left(\frac{\boldsymbol{F}_{(Bnd)}(\alpha,\gamma) - (1-\alpha)\Re(z) + \gamma\Im(z) - \alpha}{\alpha}\right)$$
$$\boldsymbol{D}_{(Bnd)}(a,b) = (1-a)F_{(Bnd)}\left(\frac{1}{1-a}, -\frac{b}{1-a}\right).$$

The theorem also has another interesting consequence, allowing us to give another characterization of the universal integral mixed spectrum.

Corollary 3.

$$\boldsymbol{M}_{(Bnd)}(z) = \sup_{bounded\ \Omega} \liminf_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^{z}(\zeta)| d|\zeta|}{\log\frac{1}{1-r}}.$$

Moreover, from the proof of the theorem, we can extract yet another description of the universal integral mixed spectrum

Corollary 4.

$$\boldsymbol{M}_{(Bnd)}(z) = \limsup_{r\to 1-} \sup_{\operatorname{diam}\Omega\le 1} \frac{\log \int_{r\mathbb{T}} |\phi'^{z}(\zeta)| d|\zeta|}{\log\frac{1}{1-r}}.$$

Some bounds for the universal integral means spectrum were obtained by Pommerenke in [17], [18]. See [20] for the list of known bounds. Some of them can be extended to the rotation case, although we are not doing it here.

The fractal approximation also allows us to describe all possible integral mixed spectra of bounded domains.

Theorem 4. *The following are equivalent:*

1. *$m(z)$ is a convex, nondecreasing in every direction, nonnegative function, $m(0) = 0$, $m(z) \le \boldsymbol{M}_{(Bnd)}(z)$, and all tangent plane to the graph of $w = m(z)$ intersect w-axes within the interval* $[0,1]$.
2. *$m(z)$ is an integral mixed spectrum of a bounded simply connected domain.*

For the classical integral means spectrum such a theorem was established in [13].

Another application of the fractal approximation is the description of the universal integral mixed spectra of unbounded domains.

Theorem 5.

$$\boldsymbol{M}_{(Unb)}(z) = \begin{cases} \boldsymbol{M}_{(Bnd)}(z), & \Re z \le 0 \\ \max\Big(\boldsymbol{M}_{(Bnd)}(z), (1+2\cos(\arg z))z - 1\Big), & \Re z > 0 \end{cases}$$

Note that the rotational part of the universal spectrum is the same for bounded and for unbounded domains.

We also obtain descriptions for the universal integral mixed spectra for m-symmetric and α-Hölder domains. In the case of the integral means spectrum such estimates were obtained by Makarov and Pommerenke in [14].

One can define the mixed spectra for non simply connected domains using the Green map instead of the Riemann map.

For regular fractals these spectra behave nicely, but even in the case of polynomial hyperbolic Julia sets some results of the simply connected case do not hold – we need to exclude some exceptional sets related to critical points.

It is not even known if we have nontrivial universal bounds.

It looks natural to try the technique of [9] to answer this question and even to establish an analogue of Theorem 3. Note that in [13] this method was used to establish fractal approximation for the dimension spectrum of non simply connected domains.

Furthermore, we believe that the universal bounds for the rotation spectrum for non simply connected domains are the same as for the simply connected (this is definitely wrong for the mixed spectrum).

2. Basic objects and their properties

In this section we introduce three types of mixed spectra. The first, *integral mixed spectrum* characterize the growth of the integral means with complex exponent of the derivative of a univalent map on $\mathbb{D}$. The second, *distortion mixed spectrum*, reflects the behavior of the derivative of a univalent map. The third spectrum, *dimension mixed spectrum*, is more geometrical and reflects the rotation of a simply connected domain in the neighborhood of a boundary point, as well as the decay of harmonic measure.

We also explore the interrelations between them.

2.1. Spectra related to the Riemann map.

Let Ω be a simply connected domain and $\phi : \mathbb{D} \to \Omega$ its Riemann mapping.

Definition 2. The *integral mixed spectrum* of the domain Ω is

$$m_\Omega(z) = \limsup_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^z(\zeta)| d|\zeta|}{\log \frac{1}{1-r}}.$$

Remark. Since Ω is a simply connected domain, we can select a branch of $\phi'^z(\zeta)$ in the domain, and the value of the limit in Definition 2 does not depend on the branch selected. It also does not depend on the Riemann map chosen.

Let us also introduce a measure of the growth of the integral mixed spectrum at infinity.

Definition 3. Let $\theta \in [0, 2\pi]$. The *characteristic of the domain Ω in the direction θ* is

$$k_\Omega(\theta) = \lim_{t\to+\infty,\ t\in\mathbb{R}} \frac{m_\Omega(te^{i\theta})}{t}.$$

The following lemma ensures the existence of the limit and describes some basic properties of the integral mixed spectrum.

Lemma 1. (1) *$m_\Omega(z)$ is a convex, nondecreasing in every direction, nonnegative function,* $m_\Omega(0) = 0$.

(2)

$$k_\Omega(\theta) = \limsup_{|z|\to 1-} \frac{\Re(e^{i\theta}\log\phi'(z))}{\log\frac{1}{1-|z|}}. \tag{2}$$

(3)

$$k_\Omega(\theta)t - 1 \le m_\Omega(te^{i\theta}) \le k_\Omega(\theta)t. \tag{3}$$

Proof. Convexity easily follows from the Hölder inequality:

$$\int_{r\mathbb{T}} |\phi'^{az_1+(1-a)z_2}(\zeta)|d|\zeta| \le \left(\int_{r\mathbb{T}} |\phi'^{z_1}(\zeta)|d|\zeta|\right)^a \left(\int_{r\mathbb{T}} |\phi'^{z_2}(\zeta)|d|\zeta|\right)^{1-a}.$$

Since $\int_{r\mathbb{T}} \log\phi'(\zeta)d|\zeta| = \log\phi'(0)$ and $\log\phi'$ is a Bloch function, the measure of $\mathrm{l}\{\zeta \in r\mathbb{T} : |\phi'(\zeta)^z| > C(\phi,z)\} \ge C_1(\phi,z)$, for some constants $C,\ C_1$, depending only on ϕ and z. So $m_\phi(z) \ge \lim_{r\to 1-} \frac{\log CC_1}{\log\frac{1}{1-r}} = 0$.

Now let $z' = az$ for some real $a > 1$. We have

$$\int_{r\mathbb{T}} |\phi'^{z'}(\zeta)|d|\zeta| \ge \int_{\zeta\in r\mathbb{T}:|\phi'(\zeta)^{z'}|>1} |\phi'^{z'}(\zeta)|d|\zeta| \ge$$

$$\int_{\zeta\in r\mathbb{T}:|\phi'(\zeta)^{z}|>1} |\phi'^{z}(\zeta)|d|\zeta| \ge \int_{r\mathbb{T}} |\phi'^{z}(\zeta)|d|\zeta| - 2\pi\,.$$

It implies that $m_\Omega(z') \ge m_\Omega(z)$.

Let $k'(\theta)$ be defined by the right part of (2). Let us prove (3) with k' instead of k. It would imply that $k'(\theta) = \lim_{t\to+\infty,\ t\in\mathbb{R}} \frac{m_\Omega(te^{i\theta})}{t} = k_\Omega(\theta)$.

The right inequality in (3) follows from

$$\int_{r\mathbb{T}} |\phi'^{z}(\zeta)|d|\zeta| \le \int_{r\mathbb{T}} \exp\left(\sup_{r\mathbb{T}} \Re(z\log\phi'(\zeta))\right) d|\zeta|$$

$\log\phi'$ is a Bloch function, so

$$\mathrm{l}\left\{\zeta \in r\mathbb{T} : \Re(z\log\phi'(\zeta)) > \sup_{r\mathbb{T}} \Re(z\log\phi'(\zeta)) - C\right\} > \frac{C_1}{1-r}$$

for some independent on r positive constants C and C_1 . So

$$\int_{r\mathbb{T}} |\phi'^{z}(\zeta)|d|\zeta| \ge \frac{C_1}{1-r} e^{-C} \exp\left(\sup_{r\mathbb{T}} \Re(z\log\phi'(\zeta))\right).$$

That implies the left inequality in (3).

□

Now let us describe a "dual" (in the sense of large deviations) to the integral mixed spectrum.

Definition 4. The *Minkowski distortion mixed spectrum* of the domain Ω is

$$d_\Omega(a,b) = \limsup_{a_1\to a, b_1\to b}\ \limsup_{r\to 1-} \frac{\log \mathrm{l}(l_{a_1,b_1})(r)}{\log\frac{1}{1-r}} + 1,$$

where

$$l_{a_1,b_1}(r) = \Bigg\{ \zeta \in \mathbb{T} : \log(\phi'(r\zeta)) \text{ lies in the same quadrant as } a+ib,$$

$$\text{and } \frac{|\log|\phi'(z)||}{\log\frac{1}{1-r}} > |a|, \frac{|arg(\phi'(z))|}{\log\frac{1}{1-r}} > |b| \Bigg\}.$$

Remark. Roughly speaking, for positive a and b, $d(a,b)$ is the Minkowski dimension of the set where $|\phi'(r\zeta)|$ grows faster then $\left(\frac{1}{1-r}\right)^a$, and $\arg\phi'(r\zeta)$ grows faster then $b\log\frac{1}{1-r}$. For other a and b the "naive" definition is just a mirror reflection of the previous description.

Also note that here, again, the definition is independent of the choice of the branch of $\arg(\phi'(z))$.

In addition, one can conclude that the value of $d_\Omega(a,b)$ does not depend on the choice of ϕ. This also follows from Lemma 3 below.

Let us summarize a few easy properties of the Minkowski distortion mixed spectrum.

Lemma 2. (1) *For any a, b either $d_\Omega(a,b) \geq 0$ or $d_\Omega(a,b) = -\infty$.*

(2) *If (a,b) lies in the same quadrant as (a_1,b_1) and, in addition, $|a| \geq |a_1|$, $|b| \geq |b_1|$, then $d(a,b) \leq d(a_1,b_1)$.*

(3) *If $|a| > 3$ or $|b| > 1$, then $d_\Omega(a,b) = -\infty$.*

(4) *$d_\Omega(0,0) = 1$, d_Ω is a concave function on the set $\{a,b : d_\Omega(a,b) > 0\}$.*

Proof. The first statement follows from the boundness of the Bloch norm of the derivative of the conformal map (so if $\mathrm{l}(l_{a_1,b_1}) > 0$, then $\mathrm{l}(l_{a_2,b_2}) > \frac{C}{1-r}$ for some $C > 0$ and $a_2 = \frac{a_1+a}{2}$, $b_2 = \frac{b_1+b}{2}$).

The second statement is evident from the definition.

The third statement follows from the standard growth estimate and Goluzin's inequality. Let us note that for a bounded Ω we can replace $|a| > 3$ by $|a| > 1$.

The fourth statement follows from Lemmas 3 (see below) and 1.

□

The integral mixed spectrum and Minkowski distortion mixed spectrum are related by a Legendre type transform. More precisely,

Lemma 3.

$$m_\Omega(z) = \sup_{a,b}\left(d_\Omega(a,b) + a\Re z + b\Im z - 1\right)$$
$$d_\Omega(a,b) = \inf_{z}\left(m_\Omega(z) - a\Re z - b\Im z + 1\right)$$

Proof. Note that

$$\int_{l_{a_1,b_1}(r)} |\phi'^z(\zeta)||d|\zeta| \geq \mathrm{l}(l_{a_1,b_1}) \max\left(\left(\frac{1}{1-r}\right)^{a_1\Re z + b_1\Im z}, 1\right).$$

It implies the " $\geq$" inequality in the first equation, and the " $\leq$" inequality in the second one.

To prove the second statement, let us note that by the theory of large deviations for Bloch martingales,

$$\log\int_{r\mathbb{T}} |\phi'^z| - a\Re z - b\Im z + 1$$

is the entropy function for the Bloch martingale $\mathcal{M}$ generated by $\log\phi'$ (see [6], Chapter II). Since $d(a,b)$ is the distribution function for the $\sum_{k=0}^n \mathcal{M}_k$, the lemma is equivalent to the large deviation property for the Bloch martingales. This fact is well known (see [12] for the discussion).

□

Let us also define the Hausdorff version of $d_\Omega(a,b)$.

Definition 5. The *Hausdorff distortion mixed spectrum* of the domain Ω is

$$\tilde{d}_\Omega(a,b) = \dim\{ld_a \cap lr_b\},$$

where dim is, as usual, the Hausdorff dimension,

$$ld_a = \begin{cases} \dim\{\zeta \in \mathbb{T} : \limsup_{r\to 1-} \frac{\log|\phi'(r\zeta)|}{\log\frac{1}{1-r}} \geq a\}, & a \geq 0 \\ \dim\{\zeta \in \mathbb{T} : \liminf_{r\to 1-} \frac{\log|\phi'(r\zeta)|}{\log\frac{1}{1-r}} \leq a\}, & a < 0 \end{cases},$$

$$lr_a = \begin{cases} \dim\{\zeta \in \mathbb{T} : \limsup_{r\to 1-} \frac{\arg\phi'(r\zeta)}{\log\frac{1}{1-r}} \geq b\}, & b \geq 0 \\ \dim\{\zeta \in \mathbb{T} : \liminf_{r\to 1-} \frac{\arg\phi'(r\zeta)}{\log\frac{1}{1-r}} \leq b\}, & b < 0 \end{cases}.$$

As usual, the Minkowski spectrum is larger.

Lemma 4.

$$\tilde{d}_\Omega(a,b) \leq d_\Omega(a,b)$$

for any simply connected domain Ω.

Proof. Let us fix a, b with $\tilde{d}_\Omega(a,b) \geq 0$. It is enough to prove the lemma only for such values of a, b. To simplify the notation, we assume that $a > 0,\ b > 0$. The proof for the other values of a, b is the same. Fix positive a_1, b_1 with $a_1 < a,\ b_1 < b$.

For every $\zeta \in ld_a \cap lr_b$ there exist a sequence $r_n(\zeta)$ such that

$$\limsup_{n\to\infty} \frac{\log|\phi'(r_n(\zeta)\zeta)|}{-\log(1-r_n)} \geq a$$

and

$$\limsup_{n\to\infty} \frac{\arg\phi'(r_n(\zeta)\zeta)}{-\log(1-r_n)} \geq b.$$

Using $\log\phi' \in \mathcal{B}$ we can assume that $r_n(\zeta) = 1 - \left(\frac{1}{2}\right)^{k_n(\zeta)}$, since $h(\left(\frac{1}{2}\right)^n, \left(\frac{1}{2}\right)^{n+1})$ is bounded independently on n.

Again from $\log\phi' \in \mathcal{B}$, we get that for large enough n, the interval of the length $C(1 - \left(\frac{1}{2}\right)^{k_n(\zeta)})$ (for some $C > 0$) with the center ζ belongs to $l_{a_1,b_1}(1 - \left(\frac{1}{2}\right)^{k_n(\zeta)})$.

So, by the covering lemma (see, e.g., Chapter 1 of [26]) for any fixed N we can select a family of non intersecting balls $B(\zeta_n, r_n)$ with the centers in $ld_a \cap lr_b$, such that their doubles (the balls with the same center and twice the radius) cover $ld_a \cap lr_b$, each ball is a subset of some $l_{a_1,b_1}(1 - \left(\frac{1}{2}\right)^n)$ for some $n > N$.

Now we take any $c > d_\Omega(a,b)$. Then, for large enough N,

$$1\left(l_{a_1,b_1}\left(1 - \left(\tfrac{1}{2}\right)^n\right)\right) \leq \left(\tfrac{1}{2}\right)^{n(1-c+\epsilon)}.$$

So $\sum r_n^c \leq \infty$, and, since N and c were arbitrary, $\tilde{d}_\Omega(a,b) = \dim ld_a \cap lr(b) \geq d_\Omega(a,b)$.

□

2.2. **Geometric spectra, rotation.** Let us now define more geometric objects, which reflect the local behavior of the boundary of a given simply connected domain.

Let Ω again be a simply connected domain and z_0 a fixed point in Ω .

The following lemma shows that the angle used in the definition of rotation is "almost" independent on the point from the δ-neighborhood of x.

Lemma 5. *If* $y_1, y_2 \in \partial\Omega_\delta \cap B(z,\delta)$ *then* $|\arg(y_1 - x) - \arg(y_1 - x)| < 2\pi$.

Here we again take the branch of the arg *with* $-\pi < \arg(z_0 - x) \leq \pi$.

Proof. Let Γ_1 and Γ_2 be two non intersecting curves joining z_0 with y_1 and y_2 respectively and lying inside Ω_δ. We can select an arc C of $\{|y-x|=\delta\}$ joining y_1 and y_2 such that $\Gamma_1 \cup \Gamma_2 \cup C$ does not contain x in the bounded component. Then $|\arg(y_1-x)-\arg(y_1-x)|$ is equal to the angular length of C, which is less than 2π.

□

Now let us define two refinements of the dimension spectra from [13].

Definition 6. The *Minkowski dimension mixed spectrum of a simply connected domain* Ω is

$$f_\Omega(\alpha,\gamma)=\lim_{\eta\to 0}\limsup_{\delta\to 0}\frac{N(\delta,\alpha,\gamma,\eta)}{\log\frac{1}{\delta}},$$

where $N(\delta,\alpha,\gamma,\eta)$ is a maximum number of disjoint balls B_n of radii δ with the centers $z_n\in\partial\Omega$ and $\delta^{\alpha+\eta}\le\omega_\Omega B_n\le\delta^{\alpha-\eta}$, $\delta^{\gamma+\eta}\le\rho(z,\delta)\le\delta^{\gamma-\eta}$.

Definition 7. The *Hausdorff dimension mixed spectrum of a simply connected domain* Ω is

$$\tilde f_\Omega(\alpha,\gamma)=\lim_{\eta\to 0}\dim E_\eta,$$

where E_η is a set of points $z\in\partial\Omega$, for which there exists a decreasing to zero sequence of radii δ_j with $\delta_j^{\alpha+\eta}\le\omega_\Omega B(z,\delta_j)\le\delta_j^{\alpha-\eta}$, $\delta_j^{\gamma+\eta}\le\rho(z,\delta_j)\le\delta_j^{\gamma-\eta}$.

Here, again, the Minkowski spectrum is larger than the Hausdorff one.

Lemma 6. *For any simply connected domain* Ω*,*

$$f_\Omega(\alpha,\gamma)\ge\tilde f_\Omega(\alpha,\gamma).$$

The proof of the lemma is practically identical to the proof of Lemma 4.

In fact, one can define yet another version of the distortion spectrum, replacing rotation of the argument by geometric rotation.

2.3. **Dimension and distortion spectra: relations.** In this section we prove Theorem 1.

Let us note that using Lemma 3 one can rewrite the Theorem as

Corollary 5.

$$f_\Omega(\alpha,\gamma)\le\inf_z\left(\alpha m_\Omega(z)+(1-\alpha)\Re(z)-\gamma\Im(z)+\alpha\right)$$

$$m_\Omega(z)\ge\sup_{\alpha,\gamma}\left(\frac{f_\Omega(\alpha,\gamma)-(1-\alpha)\Re(z)+\gamma\Im(z)-\alpha}{\alpha}\right).$$

To prove Theorem 1 we need the following technical lemma.

Lemma 7. *For all* $\zeta\in\mathbb{T}$*, if* $\phi(\zeta)=\lim_{r\to 1}\phi(r\zeta)$ *exists, then*

$$\left|\limsup_{r\to 1-}\frac{\arg\phi'(r\zeta)}{\log\frac{1}{1-r}}\right|=\left|\limsup_{r\to 1-}\frac{\arg\left(\frac{\phi(r\zeta)-\phi(\zeta)}{z}\right)}{\log\frac{1}{1-r}}\right|.$$

Proof. To prove the lemma let us consider the map $g(z)=\log(\phi(r\zeta)-\phi(z))$.

For $t<0$ let $r(t)$ be the first r for which $\Im\phi(r\zeta)=t$.

Then it is enough to prove the statement of the lemma only for $r(t)$, since for all other r we have $|\log\frac{1}{1-r}|>\log\frac{1}{1-r(t)}$.

We can join the point $g(r(t)\zeta)$ with $g(0)$ by the segment of the line $\Im z=t$ and then by some curve V non intersecting $\{g(r\zeta),\ 0\le r\le r(t)\}$. Since $g(\zeta)$ exists, V have bounded independently on $r(t)$ rotation around $g(0)$. So we can select V with bounded rotation of the tangent. So, since the total rotation of the tangent along $V \cup \{g(r\zeta),\ 0 \le r \le r(t)\}$ is 2π, $\arg g'(r(t)\zeta)$ is bounded independently on t. But $\arg g'(r(t)\zeta) = \arg\phi'(r(t)\zeta) - \arg\phi(r(t)\zeta)$, which implies that for $r = r(t)$ the right and left side of the equality of the lemma are equal up to the constant. It implies the lemma.

□

Proof of the Theorem 1. Let us fix a, b. It is enough to only consider the case $b \ge 0$.

By Theorem 3.1 from [13],

$$d_\Omega(a, 0) \ge \sup_b (1-a) f_\Omega\left(\frac{1}{1-a}, -\frac{b}{1-a}\right). \tag{4}$$

To prove the theorem we construct a domain Ω' with

$$d_{\Omega'}(a, b) = d_{\Omega'}(a, 0) = d_\Omega(a, b), \tag{5}$$

and

$$\sup_{a' \ge a} f_{\Omega'}\left(\frac{1}{1-a'}, -\frac{b}{1-a}\right) \ge f_\Omega\left(\frac{1}{1-a}, -\frac{b}{1-a}\right) \tag{6}$$

for some $\epsilon_1 > 0$.

Then the theorem will be a consequence of (4).

Let $\alpha = \frac{1}{1-a}$, $\gamma = -\frac{b}{1-a}$.

Now fix some $\eta > 0$ and let Υ be the family of balls of radius $\frac{1}{2^n}$ from the definition of $f_\Omega(\alpha, \gamma)$. Add a few balls Υ' to the family to make $\Upsilon \cap \Upsilon'$ the maximum disjoint set of balls with $\delta^{\alpha-\eta} \le \omega(B) \le \delta^{\alpha-\eta}$. Replace $\partial\Omega \cap B'$ by an analytic curve for any $B' \in \Upsilon'$. Then for any ball $B \in \Upsilon$ we have $\rho(B) \le \delta^\gamma$, $\omega(B) \ge \delta^{-eta}$.

Now we execute the construction for $n = 1, 2, \ldots$. The resulting limit domain is Ω'.

By the construction, it satisfies (6).

Also by the construction, $\Omega' \subset \Omega$, so its Riemann map ϕ' can be decomposed into $\phi \circ \psi$, where $\psi : \mathbb{D} \mapsto \mathbb{D}$ is univalent. Thus equation (5) holds because such maps ψ can not increase the Hausdorff dimension (see [19]), and because of Lemma 7.

□

Remark. The equality in the inequality of the Theorem 1 does not hold for all parameters a, b and all simply connected domains. See [13] for an example. Note, however, that for small a and b the equality always holds.

2.4. **Auxiliary results.**

Lemma 8. *Let bounded domains Ω_n have integral mixed spectra m_n. Then there exists a domain Ω with $m_\Omega(z) =$ convex hull of $\{\sup_n m_n(z)\}$.*

The Legendre-type relation from the Lemma 3 allows us to restate this for the distortion spectrum.

Corollary 6. *Let bounded domains Ω_n have Minkowski distortion mixed spectra d_n. Then there exists a domain Ω with*

$$d_\Omega(a, b) = \text{Concave hull of the set } \{\sup_n d_n(z)\}.$$

Proof. Since rescaling does not change the distortion spectrum, we can always assume that

$$\operatorname{diam} \Omega_n < \frac{1}{2^n}.$$

Let ϕ_n be a Riemann map of the domain Ω_n. Denote $\psi(z) = \sqrt{\phi(z^2)}$ be the symmetrization of the ϕ. Let Γ be the domain $\Im z \geq -\frac{1}{2}\sqrt{1-(\Re z)^2}$. Let $\Omega_n' = \psi(\Gamma \cap \mathbb{D})$. Then $d_{\Omega_n'}(a,b) = d_{\Omega_n}(a,b)$.

Moreover, all of the distortion spectrum is concentrated on $\psi(\mathbb{T} \cap \{\Re z > 0\})$.

Let us now consider the domain Ω obtained by attaching all Ω_n' to the non intersecting arcs of t_n of $\mathbb{T}$ through the "tubes" joining the central halves of the arcs with the $\psi(\partial\Gamma \cap \Re z < \frac{1}{2})$. We can select these tubes to have C^∞ tangency in all the points of the intersection with the $\mathbb{T}$ and $\partial\Omega_n'$.

Now let us consider a Riemann map ϕ of Ω . Let $\Psi_n = \phi^{-1}(\Omega_n')$ and let γ_n be a Riemann mapping of Ψ_n. Then the restriction $\phi|_{\Psi_n}$ is equal to $\psi_n \circ \gamma_n^{-1}$. Since $\gamma_n(\mathbb{T} \cap \partial\Psi_n) \subset \mathbb{T}$, the restriction $\phi|_{\Psi_n}$ has the same distortion spectrum as ψ_n. Since $\partial\Omega - \cup\partial\Omega_n$ is smooth, it implies the Lemma.

□

3. Domains with fractal boundaries

In this section we investigate a class of domains with a "nice" behavior
of the mixed spectra. We use thermodynamic formalism to investigate the behavior of the spectra and relations between them.

3.1. **Definitions, basic notation.**

Definition 8. Let Ω be a simply connected domain with the boundary $\partial\Omega$. Let ∂ be a subarc of $\partial\Omega$. ∂ is called a *Jordan repeller* if there exists a partition of ∂ into a finite number of non intersecting subarcs $\partial_1, \partial_2, \ldots, \partial_n$ (*Markov partition*) and a piecewise univalent (but not necessary onefold) map F, and for each ∂_j there exists a neighborhood $U_j \supset \partial_j$ and a univalent map $F_j : U_j \to \mathbb{C}$ such that

(1) $F|\partial_j = F_j$.
(2) (Geometry invariance) $F_j(\Omega \cap U_j) \subset \Omega$.
(3) (Boundary invariance) $F_j(\partial\Omega \cap U_j) \subset \partial\Omega$.
(4) (Markov property) The image of each ∂_j is a union of the finite set $\mathcal{A}_j$ of the arcs ∂_k:

$$F_j(\partial_j) = \cup_{k \in \mathcal{A}_j} \partial_k.$$

(5) (Expanding) $\exists n_0 : \inf_{z \in \partial} |F^{n_0}(z)| > 1$, where F^{n_0} means the n_0th derivative of F.
(6) (Mixing) For any open V with $V \cap \partial \neq \emptyset$, there exists n such that the nth iteration of F maps $V \cap \partial$ onto the whole ∂.

Remark. Note that F can be multifold only near the ends of the arcs ∂_j. For our constructions the existence of a countable exceptional set will be unimportant.

Example 1. *Mixing repeller* with respect to a map f is a Jordan curve ∂ which is F-invariant and for which $F|\partial$ is expanding. Then ∂ is a Jordan repeller (see [25]). One example of a mixing repeller is a connected polynomial hyperbolic Julia set.

Another important example is a snowflake domain, defined in the introduction.

Dynamics on the boundary of a Jordan repeller. The Markov property of the map F allows us to define a *transfer matrix* $\mathcal{D} = (\mathcal{D}_{ij})$ for the Markov partition:

$$\mathcal{D}_{ij} = \begin{cases} 1, & \text{if } j \in \mathcal{A}_i \\ 0, & \text{if } j \notin \mathcal{A}_i \end{cases}.$$

The mixing property means that for some $n > 0$, all the entries of $\mathcal{D}^n$ are positive.

We consider the space $\Sigma(\mathcal{D})$ of one-sided sequences $x = (x_1, x_2, \dots)$ consisting of numbers $1, 2, \dots, n$ with $\mathcal{D}_{x_i, x_{i+1}} = 1$ for all i. The *one-sided shift operator* S on $\Sigma\mathcal{D}$ is defined by $S(x_1, x_2, \dots) = (x_2, x_3, \dots)$.

It is known (see [12]) that there is one-to-one (up to the countable number of points) map between $\Sigma(\mathcal{D})$ and ∂ conjugating F and S. The map allows us to use the thermodynamic formalism to investigate the properties of the Jordan repeller. The correspondence is constructed as follows.

The *cylindrical sets* are the subsets of $\Sigma\mathcal{D}$ of the form $X = (x_1, x_2, \dots, x_k) = \{y \in \Sigma(\mathcal{D}) : y_i = x_i; i = 1, 2, \dots, k\}$. $k = |X|$ is a *rank* of the cylindrical subset. For every cylindrical set X, let $\partial_X = F_{x_1}^{-1} \dots F_{x_n}^{-1} \partial_{x_n}$ and by $U_X = F_{x_1}^{-1} \dots F_{x_n}^{-1} U_{x_n}$. By $\delta(X)$ we denote the diameter of U_X. By the expanding property, $\lim_{|X| \to \infty} \delta(X) = 0$, so for each $x \in \Sigma(\mathcal{D})$, the $\lim_{k\to\infty} U_{(x_1, x_2, \dots, x_k)} = \lim_{k\to\infty} \partial_{(x_1, x_2, \dots, x_k)}$ is a point, which corresponds to the point $x \in \Sigma\mathcal{D}$.

Let $\phi : \mathbb{D} \to \Omega$ be a Riemann map of Ω . It will be convenient for us to assume that $z_0 = \phi(0)$ lies in the intersection of $F^{n_0} U_i$ for some large n_0. Because of the mixing we can select this intersection to be nonempty.

The map $B = \phi^{-1} \circ F \circ \phi$ maps $\mathbb{T}$ onto itself. By the symmetry it can be extended into some neighborhood of $\mathbb{T}$. We denote $T_X = \phi^{-1} \partial_X$, $z_X = z_{T_X}$, $\zeta_X = \zeta_{T_X}$, $w_X = \phi(z_X)$. Also let $T = \phi^{-1}\partial$.

By $\omega(X)$ we denote the harmonic measure $\omega_\Omega(U_X, z_0)$. Let ρ_X be the rotation $\rho(\phi(\zeta_X), \delta(X))$.

The following Lemma connects the Riemann map ϕ and the geometry of the boundary.

Lemma 9. (1) $|\phi'(z_X)| \asymp \frac{\delta(X)}{\omega(X)}$.

(2) $|(\phi')^{-i}(z_X)| \asymp \rho(X)$.

The constants here depend only on the repeller.

Proof. The first statement is proved in [12].

To prove the second let us observe that $\rho(X) \asymp e^{\arg(\phi(z_X) - \phi(\zeta_X))}$. In view of the invariance of dynamics, $\arg \phi'(z) - \arg(\phi(z_X) - \phi(\zeta_X))$ is bounded. That implies the second statement. $\square$

Let $x \in \mathbb{T}$ and not an end of some T_X define $\psi(x) = -\log|F'(\phi(x))|$, $\chi(x) = -\log|B'(x)|$. Since the dynamics B and F are both expanding, the functions ψ and χ can be extended to the Hölder functions on the whole circle. So in a neighborhood of $\mathbb{T}$, the harmonic conjugate u to the $v = \psi - \chi$ is well-defined and Hölder on the circle (because harmonic conjugacy maps Hölder functions to Hölder). We can take $u = -\arg \frac{F'(\phi(x))}{B'(x)}$.

Let us note that by the argument principle,

$\int_{\mathbb{T}} u d\mathrm{l} = 0$.

Let us now extend the relations established in Lemma 9.

Lemma 10. *For any $x \in \phi^{-1}(U_X)$, $|X| = n$, we have*

(1) $\omega(X) \asymp \exp\left(\sum_{j=1}^{n-1} \chi(B^j x)\right)$.

(2) $\delta(X) \asymp \exp\left(\sum_{j=1}^{n-1} \psi(B^j x)\right)$.

(3) $\rho(X) \asymp \exp\left(\sum_{j=1}^{n-1} u(B^j x)\right)$.

Here the constants depend only on the Jordan repeller.

Proof. The first two relations are proven in [12].

Let us prove the third one.

By the previous lemma,

$$\log\rho(X) = \Re(-i\log(\phi'(z_X))) + \mathbf{O}(1) \tag{7}$$
$$= \arg(\phi'(B^n(B^{-n}(z_X)))) + \mathbf{O}(1) = \sum_{j=1}^{n-1} u(B^j z_X) + \mathbf{O}(1).$$

Since u is Hölder, we have for two $x_1, x_2 \in U_X$

$$\sum_{j=1}^{n-1} u(B^j x_1) - \sum_{j=1}^{n-1} u(B^j x_2) \le \sum_{j=1}^{n-1} q^n \le C.$$

It implies the last statement of the Lemma.

□

3.2. **Mixed spectra of a Jordan repeller: thermodynamics interpretation.** Here we use the standard thermodynamic formalism to describe the mixed spectra for Jordan repellers. See [2] for an extended discussion.

Let Γ be a Hölder function on $\Sigma(\mathcal{D})$ (i.e., there exists a $Q < 1$, such that $|\Gamma(x) - \Gamma(y)| \le q^n$ for all x, y with $x_i = y_i,\ (1 \le i \le n)$).

Let $S_n\Gamma(x) = \sum_{k=0}^{n-1} \Gamma(S^k x)$.

Then there exists a limit

$$P(\Gamma) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{|X|=n} e^{S_n\Gamma(x)}, \tag{8}$$

where we can take any $x \in X$. The value of $P(\Gamma)$ is called a *pressure* of Γ.

Another description of the pressure is through the *Perron-Frobenius operator.* The Perron-Frobenius operator with the weight Γ is the operator on

$$Lf(x) = \sum_{y\in T^{-1}x} f(y)e^{\Gamma(y)}.$$

$e^{P(\Gamma)}$ is a maximal simple isolated eigenvalue of the Perron-Frobenius operator.

Yet another description is given by the *variational principle.* It states that

$$P(\Gamma) = \sup_{\substack{S\text{-invariant}\\ \text{probability measure } \mu}} h_\mu + \int_{\Sigma(\mathcal{D})} \Gamma d\mu,$$

where h_μ is the entropy of the measure μ.

In the case of a Hölder continuous function there is only one measure μ_Γ for which the sup is reached. It is called the *Gibbs measure of* Γ. It satisfies the following important condition

$$\mu_\Gamma(X) \asymp e^{-P(\Gamma)n + S_n\Gamma(x)}, \quad x \in X, |X| = n. \tag{9}$$

As was shown in the previous section, we may identify the dynamics on the mixed repeller and $(\Sigma(\mathcal{D}), S)$. So we can consider the Gibbs measures on the Jordan repeller and associated measures on $\mathbb{T}$.

Let us now define the *pressure function for the Jordan repeller* ∂ by

$$p(x, y, z) = P(x\theta + yv + zu).$$

Lemma 10 and equation (9) imply that

$$e^{nP(x,y,z)} \asymp \sum_{|X|=n} \omega^x \left(\frac{\delta}{\omega}\right)^y \rho^z. \tag{10}$$

Function $p(x, y, z)$ is a real analytic on x, y, z (since it is a maximal isolated simple eigenvalue of an analytic in x, y, z family of operators). By the standard properties of pressure (see [2]), it is convex.

Let us observe (see [12]) that $p(t_0, t_0, 0) = 0$ for the $t_0 = \dim \partial = \operatorname{mdim} \partial$, $p(1, 0, 0) = 0$.

Let $\sigma(y, z)$ be a Gibbs measure on the $\mathbb{T}$ for $s(y, z)\theta + yv + zu$ and $\mu(y, z)$ be the image of the σ under ϕ. The Lebesgue measure on T is equivalent to the $\sigma_{0,0}$, so the harmonic measure on ∂ is equivalent to $\mu_{0,0}$ (see [13]).

For a B-invariant measure, μ the *entropy* h_μ is an $\int \log J_\mu d\mu$, where J_μ is the Jacobian of B with respect to μ.

Lemma 11. *For each y, z there exists a unique $s(y, z)$ such that*

$$P(s(y, z), y, z) = 0.$$

The function $s(y, z)$ is a real analytic function.

Proof. Let us fix x, y, z and consider $h(x) = P(x, y, z)$. Then, by (8),

$$h'(x) = \int \theta d\sigma_{y,z} \le P(\theta) - h_{\theta_{y,z}} = -h_{\theta_{y,z}} < 0. \tag{11}$$

Now let us observe that $P(x, y, z) \to +\infty$ if $x \to -\infty,\ y \to -\infty,\ z \to -\infty$, and $P(x, y, z) \to -\infty$ if $x \to +\infty,\ y \to +\infty,\ z \to +\infty$. So the solution of the equation $P(s(y, z), y, z) = 0$ is uniquely defined and is real analytic on y and z by the implicit function theorem.

□

Let us observe a few properties of s.

Lemma 12. (1) *$s(0, 0) = 0$.*

(2) *$s'_y(0, 0) = s'_z(0, 0) = 0$.*

(3) *The Hessian of s at $(0, 0)$ is equal to*

$$Hs(0, 0) = \begin{pmatrix} \mathcal{T}^2 & 0 \\ 0 & \mathcal{T}^2 \end{pmatrix},$$

where

$$\mathcal{T}^2 = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{j=0}^{n-1} u \circ B^j \right)^2 d\mu = \frac{1}{n} \lim_{n\to\infty} \int \left(\sum_{j=0}^{n-1} v \circ B^j \right)^2 d\mu.$$

(4) *If $\mathcal{T}^2 = 0$, then ∂ is a real analytic curve and $s \equiv 1$. Otherwise, s is strictly convex.*

Proof. The first statement follows from $p(1, 0, 0) = 0$.

To get the second one, we differentiate the equation $p(s(y,), y, z) = 0$.

We get

$$s'_y = \frac{p'_y}{p'_x} = \frac{\int_{\mathbb{T}} v d\sigma_{y,z}}{\int_{\mathbb{T}} \chi d\sigma_{y,z}}$$

and

$$s'_z = \frac{p'_z}{p'_x} = \frac{\int_{\mathbb{T}} u d\sigma_{y,z}}{\int_{\mathbb{T}} \chi d\sigma_{y,z}}.$$

Since $\sigma_{0,0} = \mathrm{l}$ and both $\int_{\mathbb{T}} v d\mathrm{l} = 0$ and $\int_{\mathbb{T}} u d\mathrm{l} = 0$ we immediately get the second statement.
Differentiating for the second time we get

$$s''_{y,y} = -\frac{(s'_y)^2 p''_{x,x} + 2s'_y p''_{x,y} + p''_{y,y}}{p'_x} \tag{12}$$

$$s''_{y,z} = -\frac{(s'_y)(s'_z)p''_{x,x} + s'_y p''_{x,z} + s'_z p''_{x,y} + p''_{z,y}}{p'_x} \tag{13}$$

$$s''_{y,y} = -\frac{(s'_z)^2 p''_{x,x} + 2s'_z p''_{x,z} + p''_{z,z}}{p'_x}. \tag{14}$$

Since p is convex and $p'_x < 0$ (see (11)), we see that s is convex.
Now let us consider $y = z = 0$. Then by (8) and because $s'_y(0,0) = s'_z(0,0) = 0$,

$$s''_{y,y}(0,0) = p''_{y,y} = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} (S_n u)^2 d\mathrm{l} \tag{15}$$

$$s''_{y,z}(0,0) = p''_{y,z} = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} (S_n u)(S_n v) d\mathrm{l} \tag{16}$$

$$s''_{z,z}(0,0) = p''_{z,z} = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{T}} (S_n v)^2 d\mathrm{l}. \tag{17}$$

Let us observe that $U = u + iv$ can be extended to an analytic function in the disc with $U(0) = 0$. So $U^2 = (u^2 - v^2) + 2iuv$ is also an analytic function with $U^2(0) = 0$. The same is true for $(S_n U)^2 = ((S_n u)^2 - (S_n v)^2) + 2i(S_n u)(S_n v)$. By the mean value theorem it means that

$$\int_{\mathbb{T}} (S_n u)^2 d\mathrm{l} = \int_{\mathbb{T}} (S_n v)^2 d\mathrm{l}$$
$$\int_{\mathbb{T}} (S_n u)(S_n v) d\mathrm{l} = 0.$$

That proves the formula for the Hessian at (0,0).
Suppose now that $\mathcal{T}^2 = 0$. Then by the result of Przytycki, Urbański, and Zdunik (see [23]), $\partial\Omega$ is an analytic curve.
Let us consider the case of $\mathcal{T}^2 \neq 0$. Suppose that for some y_0, z_0 and some direction $a = (a_1, a_2)$ we have $(Hs(y_0, z_0)a, a) = 0$. Consider the function

$$\Phi(t) = p(s(x_0, y_0) + a_1 s'_y(y_0, z_0)t + a_2 s'_z(y_0, z_0), y_0 + ta_1, z_0 + ta_2).$$

Then

$$\Phi''(0) = (Hs(y_0, z_0)a, a) = 0. \tag{18}$$

But $\Phi(t) = P(A + tB)$, where $A = s(y_0, z_0)\chi + y_0 v + z_0 u$ and $B = a_1 s'_y(y_0, z_0)t + a_2 s'_z(y_0, z_0)\chi + a_1 v + a_2 u$. So (18) implies that

$$a_1 s'_y(y_0, z_0)t + a_2 s'_z(y_0, z_0)\chi + a_1 v + a_2 u \sim 0$$

(see [2]), or $a_1 s'_y(y_0, z_0)t + a_2 s'_z(y_0, z_0)\chi \sim a_1 v + a_2 u$. It implies, in particular, that $Hs(0,0) = 0$, which gives $\mathcal{T}^2 = 0$, which is a contradiction.

$\square$

The following lemma expresses the integral mixed spectrum of the domain in terms of s.

Lemma 13. (1) $m_\Omega(z) = s(\Re z, \Im z) + \Re z - 1$.

(2) $m_\Omega(z)$ *is a real analytic function.*

(3) $m_\Omega(z) = \lim_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^z(\zeta)|d|\zeta|}{\log \frac{1}{1-r}}$.

Proof. Let us start with the proof of the third statement.

Because $\log\phi'$ is a Bloch function, and by the Lemma 9

$$\int_{(1-2^{-n})\mathbb{T}} |\phi'^z(t+i\tilde{t})|d|\zeta| \equiv 2^{-n(1-t)} \sum_{\omega(X)\sim 2^{-n}} \rho(X)^{\tilde{t}}\delta(X)^t. \tag{19}$$

Let

$$Z_n = \frac{1}{n}\log \sum_{\omega(X)\sim 2^{-n}} \rho(X)^{\tilde{t}}\delta(X)^t.$$

The third statement is equivalent to the existence of the limit $\lim_{n\to\infty} Z_n$.

The existence of the limit follows from the inequality

$$Z_{m+n} \lesssim_{q^{|n|}} Z_n + Z_m$$

(see [2]), which is equivalent to the pseudo-Markov property of harmonic measure (see [12]).

To prove the first statement of the lemma, let us observe that for an arbitrary p we can write

$$\iint_{\mathbb{D}} |\phi'(z)|(1-|z|)^p$$

as

$$\sum_{n=0}^{\infty} 2^{-n(1-t)+p} \sum_{\omega(X)\sim 2^{-n}} \rho(X)^{\tilde{t}}\delta(X)^t$$

and as

$$\sum_{n=0}^{\infty} \sum_{|X|=n} \rho(X)^{\tilde{t}}\delta(X)^t\omega(X)^{1-t+p}.$$

By the first representation and (19), the $\iint_{\mathbb{D}} |\phi''z|(1-|z|)^p < \infty$ when $p > m(t+i\tilde{t}\,) - 1$ and is infinite when $p < m(t+i\tilde{t}\,)-1$. But by the second one and (10) we get the same with $s(t,\tilde{t}\,)+t-2$ instead of $m(t+i\tilde{t}\,)-1$.

So $s(t,\tilde{t}\,)+t-2 = m(t+i\tilde{t}\,)-1$.

The second statement of the Lemma now follows from the previous Lemma.

□

In the case when F is a polynomial of degree d, J is its Julia set, and Ω is the attractor of infinity, the relation is even simpler.

Lemma 14.

$$m_\Omega(z) = -\frac{1}{\log d} P\left(\Re\left(\bar{z}\frac{\log F'\circ\phi}{\log B'}\right)\right) - 1.$$

Proof. Observe that in this case, $\chi = \log d$, so

$$p(x,y,z) = x\log d + P(yv+zu).$$

So

$$s(y,z) = -\frac{1}{\log d}\, P(yv+zu) = -y - \frac{1}{\log d}\, P\left(\Re\left(\bar{z}\frac{\log F'\circ\phi}{\log B'}\right)\right).$$

Now the Lemma follows from the previous one.

□

Recall that for a probability measure μ, $\dim\mu = \inf\{\dim E : \mu(E) = 1\}$. By the Manning formula (see [15]), $\dim\mu = \frac{h_\mu}{\int \log|B'|d\mu}$ for a B-invariant probability measure μ with positive entropy.

Let us now express the distortion spectrum in terms of the dimensions of some Gibbs measures.

Lemma 15. *Let $t = t(a,b), \tilde{t} = \tilde{t}(a,b)$ be the numbers for which*

$$d_\Omega(a,b) = m_\Omega(t+i\tilde{t}) - at - b\tilde{t} + 1.$$

Then

(1) $\dim\sigma_{t,\tilde{t}} = m_\Omega(t+i\tilde{t}) - at - b\tilde{t} + 1,$

(2) $\sigma_{t,\tilde{t}}$*-a.e.* $\lim_{r\to1}\frac{\log\phi'(r\zeta)}{\log\frac{1}{1-r}} = a+ib.$

Proof. Note that

$$m'_t(t(a,b)+i\tilde{t}(a,b)) = a, \quad m'_{\tilde{t}}(t(a,b)+i\tilde{t}(a,b)) = b. \tag{20}$$

Observe that

$$\int \log|B'|d\sigma_{t,\tilde{t}} = p'_x(t,\tilde{t}\,)$$

and

$$\begin{aligned} h(\sigma_{t,\tilde{t}}) = p(s(t,\tilde{t}\,),t,\tilde{t}\,) - \int_{\mathbb{T}}(s(t,\tilde{t})\chi + tv + \tilde{t}u)d\sigma_{y,z} = \\ - s(t,\tilde{t}\,)\int_{\mathbb{T}}\chi d\sigma_{t,\tilde{t}} - \left(t\int_{\mathbb{T}} v d\sigma_{t,\tilde{t}}\right) - \left(\tilde{t}\int_{\mathbb{T}} u d\sigma_{t,\tilde{t}}\right) = \\ - s(t,\tilde{t})p'_x(t,\tilde{t}) + tp'_y(t,\tilde{t}) + \tilde{t}p'_z(t,\tilde{t}). \end{aligned}$$

The last equation together with (20) and Manning's formula implies the first statement.

To prove the second statement, note that by Lemma 10,

$$\begin{aligned} \lim_{|X|\to\infty}\frac{\log\phi'(z_x)}{|\log(1-|z_X|)|} = \lim_{|X|\to\infty}\frac{\frac{1}{n}(S_n u(z_X) + iS_n u(z_X))}{\frac{1}{n}S_n\chi(z_X)} \\ \stackrel{\text{by ergodic theorem}}{\longrightarrow} \frac{\int v + iud\sigma_{t,\tilde{t}}}{\int\chi d\sigma_{s,t}} = \frac{p'_y + ip'_z}{p'_x} = a+ib. \end{aligned} \tag{21}$$

□

Now let us give a similar description for the dimension spectrum.

Lemma 16. *Let $t = t(1-\frac{1}{\alpha}, -\frac{\gamma}{\alpha}), \tilde{t} = \tilde{t}(1-\frac{1}{\alpha}, -\frac{\gamma}{\alpha})$ be as in the previous lemma. Then*

(1) $\dim\mu_{t,\tilde{t}} = \frac{1}{\alpha}d_\Omega(1-\frac{1}{\alpha}, -\frac{\gamma}{\alpha}) = f_\Omega(\alpha,\gamma),$

(2) *For $\mu_{t,\tilde{t}}$-almost all z* $\lim_{\delta\to1}\frac{\log\omega(B(z,\delta)) + i\log\rho(z,\delta)}{\log r} = \alpha + i\gamma.$

Proof. The second statement of the lemma is the immediate consequence of the previous lemma and Lemma 9.

The formula for the $\dim \mu$ again follows from the previous lemma and Lemma 9. Combined with the second statement it implies that

$$\frac{1}{\alpha} d_\Omega \left(1 - \frac{1}{\alpha}, -\frac{\gamma}{\alpha}\right) \le f_\Omega(\alpha, \gamma),$$

but by Theorem 1

$$\frac{1}{\alpha} d_\Omega \left(1 - \frac{1}{\alpha}, -\frac{\gamma}{\alpha}\right) \ge f_\Omega(\alpha, \gamma).$$

□

3.3. **Relations between mixed spectra for a Jordan repeller.** In this subsection we prove Theorem 2.

Proof. The first statement of the theorem follows from the Lemma 16.

The second statement of the theorem directly follows from the Lemma 13.

By Lemma 3, d and m are related by the Legendre type transform

$$d_\Omega(a, b) = \inf_z \left(m_\Omega(z) - a\Re z - b\Im z + 1\right).$$

On the other hand, by Lemma 16, $\frac{1}{\alpha} d_\Omega(1 - \frac{1}{\alpha}, -\frac{\gamma}{\alpha}) = f_\Omega(\alpha, \gamma)$. A combination of these two relations gives the third statement of the theorem.

The fourth statement is the consequence of Lemmas 15 and 16.

Since the Legendre-like transforms relating the mixed spectra of the Jordan repeller preserve the real analyticity, the fifth statement follows from the Lemma 13.

Since $m(t + i\tilde{t}) = s(t, \tilde{t}) + t - 1$, the strict convexity of $m_\Omega(z)$ when $\mathcal{T} > 0$ follows from Lemma 12. The Legendre-like relations between the universal spectra imply the concavity of f and d.

If $\mathcal{T}^2 = 0$, Lemma 12 implies that $m_\Omega(z) \equiv 0$ and $\partial\Omega$ is analytic.

This implies the last statement of the theorem.

□

4. Simply connected domains: universal spectra and fractal approximation

In this section we investigate the properties of the universal mixed spectra. We start with proving Theorem 3. Then we describe all possible integral mixed spectra by proving Theorem 4. We conclude the chapter with the description of universal spectra of some classes of plane domains.

4.1. **Fractal approximation for the universal spectra of bounded domains.** In this subsection we establish Carleson-Jones type Theorem 3.

Let us first make an observation.

Remark. Let class Υ have the following property:

$$\phi \in \Upsilon \Rightarrow \phi^*(z) = \overline{\phi(\bar{z})} \in \Upsilon.$$

All the classes we consider have this property.

Then, since $m_{\phi\mathbb{D}}(z) = m_{\phi^*(\mathbb{D})}(\bar{z})$,

$$\mathbf{M}_{(\Upsilon)}(z) = \mathbf{M}_\Upsilon(\bar{z}).$$

We also have similar symmetries for all other universal spectra.

Proof of Theorem 3. Let us fix z and let $c < \mathbf{M}_{(Bnd)}(z)$, and $C(z)$ be a large constant described later.

Pick n and consider a simply connected domain Ω_0 with $\operatorname{diam}\Omega_0 \leq 1$ and a Riemann map ϕ_0 such that $|\int_{(1-\frac{1}{n})\mathbb{T}} {\phi'}_0^z(\zeta)| > C(z)(\log n)^{2c} n^c$. Such a domain exists by the definition of the universal spectrum.

Let $\phi(z) = \phi_0\left(\left(1 - \frac{1}{n(\log n)^2}\right) z\right)$. Then we still have

$$\left|\int_{r\mathbb{T}} \phi'^z(\zeta)\right| > \frac{C(z)}{2} n^c (\log n)^{2c} \tag{22}$$

for large n, because

$$\log \phi_0' \in \mathcal{B}.$$

Let $\Omega = \phi(\mathbb{D})$. Let also

$$\Gamma = \phi\left(\left\{|z| = 1 - \frac{1}{n\log n}\right\}\right),\ \Omega_1 = \phi\left(\left(1 - \frac{1}{n\log n}\right)\mathbb{D}\right),\ \text{and } \Omega_2 = \phi\left(\left(1 - \frac{1}{n}\right)\mathbb{D}\right).$$

We also denote the Riemann maps of the corresponding domains as ϕ_1 and ϕ_2. These are simply the sections of the map ϕ.

Note that, as before, ϕ_1 satisfies (22) with some other constant $C(z)$.

To construct the desired snowflake we modify the domain Ω_1 to get a "good" polygon P for which (22) holds. Then we iterate the polygon through the snowflake construction.

Let $m = c_1 \log(n) n$ and let the points $\zeta_1, \zeta_2, \ldots, \zeta_m$ divide the circle $\left(1 - \frac{1}{n\log n}\right)\mathbb{T}$ into the equal intervals. Here c_1 is some large absolute constant. By the distortion estimates, the polygon H generated by joining the images $\phi(z_k),\ k = 1, \ldots, m$ is not self intersecting, if c_1 is large enough. Moreover, each side of the polygon

has harmonic measure comparable to $\frac{1}{n\log n}$. More precisely, for each side h_k of H

$$\frac{c_2}{n\log n} \leq \omega(h_k) \leq \frac{c_3}{n\log n}$$

for some absolute constants c_2, c_3.

Again by the distortion estimates, we can find a line L, such that $L \cap \Omega$ is an interval of the length at least $\frac{c_4}{n\log^2 n}$, and $L \cap H$ is one of the sides of H of length at least $\frac{c_5}{n\log^2 n}$, with some absolute constants c_4, c_5.

Now let I be the unit interval of L containing $L \cap H$ as the middle. Let us check that the snowflake S obtained by the iteration of $I \cup H$ satisfies $m_S(z) \geq c$.

First, let us check that S is non intersecting. Because of the self similarity, it is enough to check that the iterations of H does not intersect L. Let h_k be a side of H different from $H \cap L$. Then let us prove that the rescaled copy of Ω attached to h_k is contained in Ω. By the snowflake construction, it would imply that S is contained in Ω. Thus it does not intersect L. The length of h_k is comparable to $\frac{\phi_1'(\zeta_k)}{c_1 n\log n}$. Using the Koebe lemma and $\operatorname{diam}\Omega < 1$, we get the desired estimate, provided c_1 is large enough.

To obtain the estimate for $m_S(z)$ it is enough to check that

$$\lim_{k\to\infty} \frac{\int_{r_k\mathbb{T}} |{\phi'}_S(\zeta)^z| d\zeta}{\log \frac{1}{1-r_k}} \geq c,$$

where $r_k = 1 - \frac{1}{n^k \log^k n}$ and ϕ_S is the Riemann map of S.

Let us consider the k-th generation of the snowflake construction. It consists of $(m+2)^k$ copies of H. By the pseudo Markov property of harmonic measure on the snowflake (see [11]), the harmonic measure of each of them is $\sim_{q^k}$-equivalent to $\frac{1}{c_1^k n^k \log^k n}$. On the other hand, the size of the piece corresponding to the cylinder $X = (x_1, x_2, \ldots, x_k)$ is $\sim_{q^k}$-equivalent to $\prod_{j=1}^k \phi_1'(\zeta_k) \frac{1}{c_1^k n^k \log^k n}$.

It implies that

$$\int_{r_k \mathbb{T}} |\phi'(\zeta)^z| d\zeta \sim_{q^k} \int_{r_{k-1}\mathbb{T}} |\phi'(\zeta)^z| d\zeta \int_{(1-\frac{1}{n})\mathbb{T}} |\phi'(\zeta)^z| d\zeta$$

which gives us the theorem.

□

Now let us prove the corollaries of the Theorem.

Proof of the Corollary 1. The first statement is a consequence of the fractal approximation (Theorem

3), the equality of the Minkowski and Hausdorff dimension spectra for the fractals (Theorem 2), and the Legendre type relation between distortion and integral spectra (Lemma 3). We also need to use the fact that for general domains $\tilde{d}_\Omega(a,b) \le d_\Omega(a,b)$ (Lemma 4).

The second equality again follows from the fractal approximation (Theorem

3), the corresponding equality for the fractals (Theorem 2), and the inequality between the distortion and dimension spectra (Theorem 1).

□

Proof of the Corollary 2. All the statements of the corollary follow from the previous corollary and Theorem

2.

□

Proof of the Corollary 3. It immediately follows from the fractal approximation and the existence of the limit $\lim_{r\to 1-} \frac{\log \int_{r\mathbb{T}} |\phi'^z(\zeta)| d|\zeta|}{\log \frac{1}{1-r}}$ for the snowflakes. □

Proof of the Corollary 4. Let

$$c < \limsup_{r\to 1-} \sup_{\Omega} \frac{\log \int_{r\mathbb{T}} |\phi'^z(\zeta)| d|\zeta|}{\log \frac{1}{1-r}}.$$

Then there exists a domain Ω ($\operatorname{diam} \Omega \le 1$) for which

$$\int_{r\mathbb{T}} |\phi'^z(\zeta)| d|\zeta| > C(z) \log \frac{1}{1-r}^{2c} \left(\frac{1}{1-r}\right)^c.$$

So, as in the proof of Theorem 3, we can construct a snowflake Ω_1 with $m_{\Omega_1} \ge c$.

It implies that

$$\mathbf{M}_{(Bnd)}(z) \ge \limsup_{r\to 1-} \sup_{\Omega} \frac{\log \int_{r\mathbb{T}} |\phi'^z(\zeta)| d|\zeta|}{\log \frac{1}{1-r}}.$$

But the inequality in the other direction is evident.

□

4.2. **Characterization of all possible mixed spectra.** In this subsection we prove Theorem 4.

In view of the Legendre-like relation from Lemma 3, the Theorem is equivalent to the corresponding fact for the Minkowski distortion mixed spectrum.

Corollary 7. *The following are equivalent:*

(1) $d(0,0) = 1$, $d(a,b) \le \boldsymbol{D}_{(Bnd)}(a,b)$, *for any* a, b *either* $d(a,b) \ge 0$ *or* $d(a,b) = -\infty$, $d(a,b)$ *is a concave function on the set* $\{(a,b) : d(a,b) \ge 0\}$.

(2) *$d(a,b)$ is a Minkowski distortion mixed spectrum of a bounded simply connected domain.*

Proof of Theorem 4. In fact, instead of the theorem, it will be easier for us to prove Corollary 7 (which, as was mentioned above, is equivalent to the theorem).

The implication 2. $\Rightarrow$ 1. follows from Lemma 2

Now let us prove the implication 1. $\Rightarrow$ 2.

First we fix a pair a_0, b_0 for which $\mathbf{D}_{(Bnd)}(a,b) \geq 0$ and a nonnegative $c \leq \mathbf{D}_{(Bnd)}(a,b)$ and construct a domain Ω with

$$d_\Omega(a,b) = \begin{cases} 1 - \frac{a}{a_0} + \frac{a}{a_0}c, & \text{if } 0 \leq a \leq a_0, b \leq \frac{b_0}{a_0}a, \\ -\infty, & \text{otherwise.} \end{cases}$$

For this let us pick a snowflake domain Ω_1 with $dm_{\Omega_1}(a_0,b_0) \geq c$. Such a domain exists by Corollary 1.

Let ϕ_1 be a Riemann map of the domain, and let B be the corresponding dynamics in the neighborhood of $\mathbb{T}$. By the discussion in Section 3, there exists a B-invariant set $E \subset \mathbb{T}$ of the dimension c such that $\lim_{r\to 1-} \frac{\log {\phi'}_1(r\zeta)}{\log\frac{1}{1-r}} = a_0 + ib_0$. In fact, we can take any B-invariant subset of the dimension c of the set $\operatorname{supp}\sigma_{y,z}$ for the corresponding y and z. $\mathbb{T} - E = \cup_k I_k$ for non intersecting arcs I_k. Since E is B-invariant, we can select a subfamily $\mathcal{I} = I^1, I^2, \dots, I^m$ of the arcs I_k such that for any k $B^l(I_k) \in \mathcal{I}$ for some l. For each I_k let $\operatorname{rnk}(I_k) = \min_l B^l(I_k) \in \mathcal{I}$. Let us attach to the arc I_k an analytic curve A_k, which lies inside $\mathbb{D}$, has the C^∞ tangency with the $\mathbb{T}$ at the endpoints of I_k, and such that $\forall \theta \in A_k \ : \ |\theta| \geq 1 - 2^{-2^{\operatorname{rnk}(I_k)}}$

. Let D_1 is the subdomain of $\mathbb{D}$ bounded by $\cup_k A_k$ and E. Let $\Omega = \phi_1(D_1)$. Let us prove that Ω is the domain with the desired spectrum.

Let ψ be a Riemann map of D_1. By the conformality criterion from [24], the map ψ is conformal in all the points of $\mathbb{T}$. So $\dim \psi^{-1}E = \dim E$.

The Riemann mapping of Ω is given by $\phi = \phi_1 \circ \psi$. Then

$$\phi'(r\zeta) = {\phi'}_1(\psi(r\zeta))\psi'(r\zeta).$$

Since ψ is conformal up to the boundary, the $\sup_r |\phi'(r\zeta)| < \infty$ for $\zeta \notin \psi^{-1}E$, and

$$\lim_{r\to 1-} \frac{\log \phi'(r\zeta)}{\log\frac{1}{1-r}} = a_0 + ib_0$$

for $\zeta \in \psi^{-1}E$. It implies that

$$\tilde{d_\Omega}(a_0,b_0) = \begin{cases} 1, & (a,b) = (1,0) \\ c, & (a,b) = (a_0,b_0) \\ -\infty, & \text{otherwise.} \end{cases}$$

But the sets E and $\phi^{-1}E$ are Cantor-like, there Hausdorff and Minkowski dimensions coincides (see [7]). It implies the required statement for $\tilde{d}$.

Now let us consider a general $d(a,b)$ satisfying the conditions from the first part of the Corollary 7. Let us select the sequence of triples $A = \Big\{(a_n, b_n, c_n)\Big\}$ such that the graph of d is a concave span of the set A and the point $(0,0,1)$.

By the above construction, for any n there exists a bounded simply connected domain Ω_n with

$$d_{\Omega_n}(a,b) = \begin{cases} 1 - t + tc_n, & \text{if } 0 \leq a \leq a_n, b \leq \frac{b_n}{a_n}a, \\ -\infty, & \text{otherwise.} \end{cases}$$

Now we use Lemma 8 to create the domain with $d_\Omega(a,b) = d(a,b)$.

□

4.3. **Universal mixed spectra for different classes of domains.** We start with the proof of Theorem 5.

Proof of Theorem 5. The function $\phi_\alpha(z) = (1-z)^{-2\cos\alpha e^{i\alpha}} (0 \le \alpha < \frac{\pi}{2})$ is univalent in the unit disc and maps it onto an unbounded logarithmic spiral Ω_α. $\phi_\alpha'(z) = -2\cos\alpha e^{i\alpha}(1-z)^{-e^{2i\alpha}+2}$. So, by the second statement of Lemma 1,

$$k_{\Omega_\alpha}(\theta) = \max\left(0, -\cos(\theta+2\alpha) + 2\cos\theta\right).$$

If $0 \le \theta < \frac{\pi}{2}$, take $\alpha_\theta = \frac{\pi}{2} - \frac{\theta}{2}$.

Then

$$\mathbf{M}_{(Unb)}(te^{i\theta}) \ge k_{\Omega_{\alpha_\theta}}(\theta) = 1 + 2\cos\theta.$$

So, since $\mathbf{M}_{(Unb)}(z) = \mathbf{M}_{Unb}(\bar{z})$ by Remark at the beginning of the Chapter, it is enough to check that

$$\mathbf{M}_{(Unb)}(z) \le \begin{cases} \mathbf{M}_{(Bnd)}(z), & \Re z \le 0 \\ \max\Big(\mathbf{M}_{(Bnd)}(z), (1+2\cos(\arg z))z - 1\Big), & \Re z > 0 \end{cases}.$$

To get the inequality we combine the proof of Theorem 5.4 from [13] and Lemma 7.

□

Theorem 4 allows us to characterize the universal spectra for some other classes of domains.

Let us first consider the class (α-Hölder) of all bounded α-Hölder domains, that is, the domains for which the Riemann map ϕ is α-Hölder in the whole $\overline{\mathbb{D}}$.

Lemma 17. *$\boldsymbol{M}_{(\alpha\text{-Hölder })}(z)$ is the convex hull of $\boldsymbol{M}_{(Bnd)}(z)$ and the line $w = (1-\alpha)x + c_0$, where $c_0 = \sup\{c : (1-\alpha)x + c \le \boldsymbol{M}_{(Bnd)}(x,0)\ \forall x\}$.*

Proof. The necessary and sufficient condition for a domain to be α-Hölder is

$$|\phi'(z)| \le C(1-|z|)^{\alpha-1}$$

(see [5]). So for an α-Hölder domain, $k_\phi(0) \le (1-\alpha)$. That implies the upper estimate for $\mathbf{M}_{(\alpha\text{-Hölder })}(z)$.

On the other hand, if $k_\phi(0) < 1-\alpha$, then the domain is α-Hölder .

By Theorem 4, for any $\epsilon > 0$ there exists a domain Ω with m_Ω equal to the convex hull of $\mathbf{M}_{(Bnd)}$ and the line $(1-\alpha-\epsilon)x + c_0$. The Ω is α-Hölder , which implies the lower estimate for $\mathbf{M}_{(\alpha\text{-Hölder })}(z)$.

□

Note that the Hölder restriction did not affect the rotational part of the universal spectrum.

Let us now consider the class S^M of domains Ω with $\operatorname{mdim}\partial\Omega \le M$. Here, again, the dimensional restriction does not affect the rotational part of the spectrum.

Lemma 18. *For $1 \le M < 2$, the universal spectrum $\boldsymbol{M}_{(S^M)}(z)$ is the convex hull of $\boldsymbol{M}_{(Bnd)}(z)$ and the point $(M, 0, M-1)$.*

Proof. By Theorem 3.1 from [13], $m(\operatorname{mdim}\partial\Omega) \le \operatorname{mdim}\partial\Omega - 1$. It implies the upper estimate on $\mathbf{M}_{(S^M)}(z)$.

On the other hand, by the Theorem 4, if $\mathbf{M}_{(\alpha\text{-Hölder })}(M,0) \ge M-1$, there exists a domain Ω_α with the integral mixed spectrum m_{Ω_α} equal to the convex hull of $\mathbf{M}_{(\alpha\text{-Hölder })}(z)$ and the point $(M, 0, M-1)$.

The domain Ω_α is α-Hölder , so, by Theorem 3.1 from [13], $\dim\partial\Omega_\alpha = M$. The lemma now follows from $\lim_{\alpha\to 1}\mathbf{M}_{(\alpha\text{-Hölder })}(z) = \mathbf{M}_{(Bnd)}(z)$.

□

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