Real Analysis I
Assignment 1, due September 30

Problem 1 of 5.

Let $\mathcal{E}$ be a collection of n subsets of a set X. Prove that $\mathcal{M}(\mathcal{E})$ is finite. What is the maximal number of elements of $\mathcal{M}(\mathcal{E})$?

Problem 2 of 5.

Let $\mathcal{M}$ be an infinite σ-algebra. Show that ...

1. $\mathcal{M}$ contains an infinite sequence of disjoint non-empty sets;
2. $\mathcal{M}$ is uncountable.

Problem 3 of 5.

Let $(X, \mathcal{M}, \mu)$ be a measure space, and consider a sequence $\{E_j\}_{j\geq 1}$ in $\mathcal{M}$. Define

$$\liminf E_j = \bigcup_{k\geq 1}^{\infty} \bigcap_{j=k}^{\infty} E_i \,, \qquad \limsup E_j = \bigcap_{k\geq 1}^{\infty} \bigcup_{j=k}^{\infty} E_i \,.$$

1. Show that
$$\liminf E_j = \{x : x \in E_j \text{ for all but finitely many } j\} \,,$$
$$\limsup E_j = \{x : x \in E_j \text{ for infinitely many } j\} \,.$$
Conclude that $\liminf E_j \subset \limsup E_j$.
2. Give an example of a sequence $\{E_j\}$ where $\liminf E_j \neq \limsup E_j$.
3. If the limits supremum and infimum are equal, then their common value is called the *limit of the sequence of sets* E_j. Prove that the limit of a sequence of sets E_j exists if and only if the sequence of characteristic functions of the E_j has a limit.
4. Show that $\mu(\liminf E_j) \leq \liminf \mu(E_j)$.
If $\mu(\bigcup_{j=1}^{\infty} E_j) < \infty$, then also $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.

Problem 4 of 5 (*The Borel-Cantelli lemma).*

Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose $(E_j)_{j\geq 1}$ is a sequence of measurable sets with the property that

$$\sum_{j=1}^{\infty} \mu(E_j) < \infty \,.$$

Show that $\mu(\limsup E_j) = 0$, i.e. for μ-a.e. x $\exists n(x)$: $\forall j > n(x)$, $x \notin E_j$.

Problem 5 of 5.

Let μ^* be an outer measure on X induced from a premeasure μ_0, and let μ be the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets.

1. Prove that $\quad \mu^*(A) = \inf_{E \in \mathcal{M} : E \supset A} \mu(E) \quad$ for all $A \subset X$.

2. If $\mu_0(X) < \infty$, define the *inner measure* of a set $A \subset X$ by $\mu_*(A) = \mu_0(X) - \mu^*(A^c)$. Prove that A is measurable, if and only if $\mu^*(A) = \mu_*(A)$.