# Real Analysis I
# Assignment 2, due October 7

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**Problem 1 of 5.**

Given a set $E \subset \mathbb{R}$, consider the open sets

$$U_n = \left\{ x \in \mathbb{R} : d(x, E) = \inf_{y \in E} |x - y| < \tfrac{1}{n} \right\}.$$

1. If $E$ is compact, prove that $m(E) = \lim m(U_n)$.
2. Give an example of a bounded open set where the conclusion fails.
   **Hint:** It might help to construct a nowhere dense set of positive measure.

**Problem 2 of 5.**

Let $f$ be a measurable real-valued almost everywhere finite function on a measure space $(X, \mathcal{M}, \mu)$ with a probability measure $\mu$. Prove that there exists a right continuous increasing function $g$ on interval $[0,1]$ which is *equimeasurable* with $f$, i.e.

$$\mu \{x : \ f(x) > a\} = m \{t : g(t) > a\} \text{ for all } a \in \mathbb{R}.$$

**Problem 3 of 5.**

Let $f(x)$ be a continuous function on $[0,1]$. Let

$$E := \{x : f'(x) \, \text{exists}\}.$$

Prove that $E$ is Lebesgue measurable.

**Problem 4 of 5.**

Construct a Borel measurable function that assumes every value in $[-\infty, \infty]$ on each nonempty subinterval of $(0,1)$.

**Problem 5 of 5 (*Absolute continuity and tightness of Lebesgue integral)*.**

Let $f$ be complex-valued measurable function on $(X, \mathcal{M}, \mu)$ with a $\sigma$-finite $\mu$. Assume that $\int_X |f(x)| \, d\mu(x) < \infty$. Show that

$$\forall \epsilon > 0, \ \exists \delta > 0 \text{ such that if } \mu(E) < \delta \text{ then } \int_E |f(x)| \, d\mu(x) < \epsilon$$

and

$$\forall \epsilon > 0, \ \exists E, \ \mu(E) < \infty \text{ such that } : \int_{E^c} |f(x)| \, d\mu(x) < \epsilon.$$