Real Analysis I
Assignment 3, due October 14

Problem 1 of 5.

Let $(f_n)_{n\geq 1}$ be a sequence of measurable almost everywhere finite real-valued functions on $(X, \mathcal{M}, \mu)$, where μ is a σ-finite measure. Prove that there exist constants $c_n > 0$ such that the series $\sum c_n f_n(x)$ converges for μ-a. e. $x \in X$.
Hint: Prove it for a finite μ first.

Problem 2 of 5.

1. Let f be a continuous function on a closed $K \subset \mathbb{R}$. Show that there exists a continuous function F on $\mathbb{R}$ such that $F(x) = f(x)$ if $x \in K$.

2. Prove that every measurable function f on $[a, b]$ is a Lebesgue almost everywhere limit of a sequence (f_n) of continuous functions.

3. Is it always possible to choose this sequence to be monotone?

Problem 3 of 5 (*The Nikodym distance*).

Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

1. Verify that $d(A, B) := \mu(A \triangle B)$ defines a metric on $\mathcal{M}/\sim$ with a suitable equivalence relation $\sim$. (Be brief, but precise!)

2. Prove that the metric space is complete.

Problem 4 of 5.

Does there exist a dense subset of $\mathbb{R}^n$, $n \geq 2$ such that no three points in it are collinear?

Problem 5 of 5 (*Criterium for convergence in* $L^1(\mu)$ *with a* σ*-finite* μ).

Let μ be a σ-finite measure and $(f_n)_{n\geq 1}$ and f be measurable functions. Show that $f_n \to_{L^1(\mu)} f$ iff all of the following three conditions are satisfied

1. f_n converges to f in measure.

2. The sequence (f_n) is *uniformly integrable*, i.e.

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that if } \mu(E) < \delta \text{ then } \int_E |f_n(x)|\, d\mu(x) < \epsilon \text{ for any } n.$$

3. The sequence (f_n) is *uniformly tight*, i.e.

$$\forall \epsilon > 0,\ \exists E,\ \mu(E) < \infty \text{ such that for any } n: \int_{E^c} |f_n(x)|\, d\mu(x) < \epsilon.$$

Remark: Note that if μ is finite, the last condition is automatically satisfied.