Real Analysis I
Assignment 4, due November 4

Problem 1 of 5.

1. Prove that the iterated integrals corresponding to the double integral
$$\int_0^\infty \int_0^\infty e^{-xy} \sin x \sin y \, dx \, dy$$
both exist and their values coincide. Does the double integral exist?

2. Prove that the iterated integrals corresponding to the double integral
$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dx \, dy$$
both exist, but their values do not coincide.

3. Prove that the iterated integrals corresponding to the double integral
$$\int_{-1}^1 \int_{-1}^1 \frac{xy}{(x^2 + y^2)^2} \, dx \, dy$$
both exist and their values coincide, but the double integral does not exist.

Problem 2 of 5.

Let M be a positive definite symmetric $n \times n$ matrix. Find the measure of the ellipsoid
$$E = \{x \in \mathbb{R}^n : x \cdot Mx < 1\}$$
in terms of M and the measure of the unit ball.

Problem 3 of 5.

Let ν be a complex measure on a measurable space $(X, \mathcal{M})$.

1. Show that there exists unique measure μ such that for each measurable $E \subset X$,
$$\mu(E) = \sup \left\{ \sum_{j=1}^n |\nu(E_j)| : \bigsqcup_{j=1}^n E_j = E \text{ and } E_j \in \mathcal{M} \text{ for all } 1 \le j \le n \right\}.$$
μ is called the *total variation* of ν and denoted by $|\nu|$.

2. Show that $\nu \ll |\nu|$ and that $|\nu|$-a.e.
$$\left| \frac{d\nu}{d|\nu|} \right| = 1.$$
Show that $|\nu|$ is the unique measure satisfying these conditions.

Problem 4 of 5.

Given an arbitrary countable set $K \subset \mathbb{R}$, construct a strictly increasing function on $\mathbb{R}$ whose set of discontinuities is exactly K.

Problem 5 of 5 (*Helly's Theorem*).

Let $\{f_n\}$ be a sequence of functions of bounded variations on some fixed segment $[a,\, b]$. Suppose that for some constant $K > 0$ and for all $n \in \mathbb{N}$,

$$|f_n(a)| \leq K \text{ and } \mathbf{Var}_{[a,\,b]} f_n \leq K.$$

Show that there exists a subsequence f_{n_k} converging at every $x \in [a,\, b]$.
Hint: Prove it first for continuous increasing f_n.