Real Analysis I
Assignment 5, due November 11

Problem 1 of 5.

Let $f \in L^1(\mathbb{R}^n)$, not identically zero. Prove that $Hf(x) \notin L^1(\mathbb{R}^n)$.

Problem 2 of 5.

Let $z : [0, 1] \to \mathbb{C}$ be a *rectifiable Jordan curve* in $\mathbb{C}$, i.e. an injective map such that

$$\sup_P \sum_{j=1}^{n} \big(|z(t_j) - z(t_{j-1})|\big) < \infty,$$

where the supremum is taken over all partitions $P = \{0 = t_0 < t_1, \cdots < t_n = 1\}$ of $[0, 1]$.
Show that there is a *re-parametrization* (i.e. an increasing function $t(s) : [0, 1] \to [0, 1]$) such that $z(t(s))$ has a tangent vector for almost every $s \in [0, 1]$.

Problem 3 of 5.

Let $f : \mathbb{R} \to \mathbb{R}$. Prove that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq M|x - y|$$

for some M and all $x,\ y \in \mathbb{R}$, if and only if f satisfies the following two properties:

1. f is absolutely continuous.
2. $|f'(x)| \leq M$ for a.e. x.

Problem 4 of 5.

Prove that there does not exist a Lebesgue-measurable subset E of the real line $\mathbb{R}$ such that for any $1 > a > 0$

$$m\left([0,\, a] \cap E\right) = \frac{a}{2}$$

Problem 5 of 5.

Prove that for each $0 < p \leq 1$ the space of sequences

$$l^p = \left\{ \mathbf{x} = (x_n)_{n \geq 1} \,:\, x_n \in \mathbb{R},\ \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

form a metric vector space with invariant metric

$$d(\mathbf{x}) = \sum_{n=1}^{\infty} |x_n|^p.$$

Prove that it is complete.
For which values of p is it a normed space?