Real Analysis I
Assignment 6, due November 23

Problem 1 of 5.

Let X be a Banach space. Show that the set of *invertible operators* (i.e. such bijective operators $T \in L(X,X)$ that $T^{-1} \in L(X,X)$) is open in $L(X,X)$.

Problem 2 of 5.

Let M be a subspace of a normed vector space X. Show that M^* is isometric to a quotient space of X^*.
Hint: Use Hahn-Banach Theorem.

Problem 3 of 5.

Let X be a real vector space, and let $K \subset X$ be a convex set such that $0 \in K$ and

$$\forall x \in X \exists t > 0 \text{ such that } x/t \in K.$$

Let us define the Minkowski functional of K by

$$p_K(x) = \inf \{t > 0 \,:\, x/t \in K\}$$

1. Prove that p_K is a convex function.
2. Show that if K is also *symmetric* (i.e. $x \in K$ iff $-x \in K$), then p_K is a seminorm.
3. Show that any seminorm is the Minkowski functional of some convex symmetric K.
4. Show that p_K is a norm iff K is symmetric and does not contain any subspace of X.

Problem 4 of 5.

Show that any seminorm p on a real vector space can be represented as

$$p(x) = \sup_{s \in S} f_s(x),$$

where $(f_s)_{s \in S}$ is a family of linear functionals on X.

Problem 5 of 5.

Give an example of complete metric space X and a sequence of closed embedded balls $B_{j+1} \subset B_j$ such that $\cap_j B_j = \emptyset$.