# Real Analysis I
# Assignment 7, due November 25

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**Problem 1 of 5.**

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space $X$ such that $\|\cdot\|_1 \leq \|\cdot\|_2$. Show that if $X$ is complete with respect to both norms, then for some $\alpha > 0$, $\|\cdot\|_1 \geq \alpha\|\cdot\|_2$.

**Problem 2 of 5.**

Let $X$ be a Banach space, $(x_n)$ be a sequence of elements of $X$ converging, in norm, to $x \in X$, and $(f_n)$ be a sequence of elements of $X^*$ weak$^*$-converging to $f \in X^*$. Show that $\lim_{n\to\infty} f_n(x_n) = f(x)$.

**Problem 3 of 5.**

Let $(x_n)$ be a sequence of elements of a Hilbert space $X$ which weakly converge to $x \in X$. Assume also that $\limsup \|x_n\| \leq \|x\|$. Show that $\|x_n - x\| \to 0$.

**Problem 4 of 5.**

Assume that $X$ is an infinite-dimensional Hilbert space. Prove that ...

1. Every orthonormal sequence in $\mathcal{H}$ converges weakly to zero;
2. The unit sphere $S = \{x \in X : ||x|| = 1\}$ is weakly dense in the closed unit ball $\bar{B} = \{x \in X : ||x|| \leq 1\}$, i.e, every $x \in \bar{B}$ is the weak limit of a sequence in $S$.

**Problem 5 of 5.**

Let $X$ be a Hilbert space. The **adjoint** of a linear transformation $T : X \to X$ is defined by the property that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for all } x, y \in X\,.$$

Clearly, $(T^*)^* = T$.

1. Show that the ranges and nullspaces of $T$ and $T^*$ are related by
$$\mathcal{R}(T)^\perp = \mathcal{N}(T^*)\,, \quad \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}\,.$$
*Remark:* In particular, if $T$ is **self-adjoint** ($T^* = T$), then $\mathcal{N}(T)$ and $\overline{\mathcal{R}(T)}$ are complementary orthogonal subspaces.
2. Let $P$ be the orthogonal projection onto a closed subspace $V \subset X$, i.e., for each point $x \in X$, its image $Px$ is the point of $V$ closest to $x$. Verify that $P$ is self-adjoint, and that $P^2 = P$.
3. Conversely, if $P$ is self-adjoint and $P^2 = P$, show that $V = \mathcal{R}(P)$ is closed and $P$ is the orthogonal projection onto $V$.