Real Analysis I
Assignment 8, due December 2

Problem 1 of 5 (*The Mean Ergodic Theorem*).

Let U be a unitary operator on the Hilbert space H, $M = \{x : Ux = x\}$ and P the orthogonal projection onto M. Let

$$S_n = n^{-1} \sum_{j=0}^{n-1} U^j.$$

Show that for all $x \in H$, $\lim_{n\to\infty} \|S_n x - Px\| = 0$.

Problem 2 of 5.

Prove that $L^p(\mathbb{R}^n)$ is separable for $1 \leq p < \infty$ but not for $p = \infty$.

Problem 3 of 5.

Suppose that $f \in L^p \cap L^\infty$ for some $1 < p < \infty$. Show that $f \in L^q$ for all $q \geq p$ and

$$\lim_{q\to\infty} \|f\|_q = \|f\|_\infty.$$

Problem 4 of 5.

Suppose $0 < p_0 < p_1 \leq \infty$. Find examples of functions on $\mathbb{R}^+$ (with Lebesgue measure) such that $f \in L^p$ if and only if ...

1. $p_0 < p < p_1$;
2. $p_0 \leq p \leq p_1$;
3. $p = p_0$.

Hint: Try functions of the form $f(x) = x^{-\alpha} |\log x|^\beta$

Problem 5 of 5.

Let (X, d) be a metric space with $H_\alpha(X) > 0$. Show that H_β is not σ-finite on X for any $0 < \beta < \alpha$.