Löwner evolution: an introduction.

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A bit more complex analysis:

$X \subset \mathbb{C}$ is locally-connected if $\forall \varepsilon > 0\ \exists \delta > 0: \forall x, y: |x-y| < \delta \Rightarrow \exists X_1 \subset X$-connected, $x, y \in X_1$, $|X_1| < \varepsilon$.

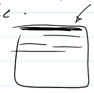
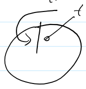
Example (non-l.c.) 

Thm (Caratheodory) The conformal map $f: \mathbb{D} \to \Omega$ has a continuous extension on $\overline{\mathbb{D}} \iff \partial\Omega$ is locally connected.

2) Ω is a Jordan domain (i.e. $\partial\Omega$ is Jordan curve) iff f extends to a homeomorphism $\overline{\mathbb{D}} \to \overline{\Omega}$.

In general, let Ω - s.c., $w_0 \in \Omega$. Consider Caratheodory metric on Ω: $d_C(z_1, z_2) = \inf\{|\gamma|$, γ - closed loop or crosscut separating z_1 and z_2 from $w_0\}$. $d_C(w_0, z) := \lim_{w \to w_0} d_C(w, z)$.

The caratheodory compactification of Ω, $\hat{\Omega}$ is a completion of Ω in this metric.

Examples:  ,  $f(\cdot) \in \partial\hat{\Omega}$; two points in $\partial\hat{\Omega}$.

Then

1) $\hat{\Omega} = \overline{\Omega} \iff \partial\Omega$ is l.c.

2) f always extends to $f: \overline{\mathbb{D}} \to \hat{\Omega}$, f^{-1} is $\frac{1}{2}$-Hölder (Bearling's Thm).

Half-plane capacity:

$A \subset \mathbb{H} = \{\operatorname{Im} z > 0\}$ is called compact hull if 1) $\overline{A}$ is compact, 2) $A = \overline{A} \cap \mathbb{H}$, 3) $\mathbb{H} \setminus A$ - simply connected.

Main example: curve γ from 0 to ∞ generates a sequence of compact hulls.

Fact: $\exists!\ g_A: \mathbb{H} \setminus A \to \mathbb{H}$: $\lim_{z \to \infty} (g_A(z) - z) = 0$ (Hydrodynamic normalization).

Pf. Take any map with $g(\infty) = \infty$, and normalize + reflection.

Def $\operatorname{hcap} A := \lim_{z \to \infty} z(g_A(z) - z)$, i.e. $g_A(z) = z + \frac{\operatorname{hcap} A}{z} + \ldots$

Remark. If $A \subset \mathbb{C}$ - compact, $\mathbb{C} \setminus A$ - connected, $f_A: \mathbb{D}_- \to \mathbb{C} \setminus A$, $f_A(\infty) = \infty$, $f'(\infty) > 0$ ($f_A(z) = b_{-1} z + b_0 + \frac{b_1}{z} \ldots$), then $b_{-1} = \operatorname{cap} A$.

Properties of $\operatorname{hcap}(A)$:

1) $\lambda > 0 \Rightarrow \operatorname{hcap}(\lambda A) = \lambda^2 \operatorname{hcap} A$

Pf $g_{\lambda A}(z) = \lambda\, g_A(z/\lambda)$ ∎

2) A, A' - compact hulls, $A \subset A'$. Then

$\operatorname{hcap} A' = \operatorname{hcap} A + \operatorname{hcap}(g_A(A' \setminus A))$.

Pf. $g_{A'} = g_{g_A(A' \setminus A)} \circ g_A$, expand at ∞ ∎

3) For $x \in \mathbb{R}$, $\operatorname{hcap}(A + x) = \operatorname{hcap} A$

Pf $g_{A+x}(z) = g_A(z - x) + x$ ∎

Examples. 1) $A = \mathbb{D} \cap \mathbb{H}$. $g_A(z) = z + \frac{1}{z}$, $\operatorname{hcap} A = 1$.

2) $A = [0, i] \Rightarrow g_A(z) = \sqrt{z^2 + 1} = z + \frac{1}{2z} + \ldots \Rightarrow \operatorname{hcap} A = \frac{1}{2}$.

Lemma $\operatorname{hcap} A \geq 0$, $\operatorname{hcap} A = 0 \iff A = \emptyset$.

Pf. Let $v(z) := \operatorname{Im}(z - g_A(z))$. Then $\lim_{z \to \infty} v(z) = 0$, $v(z) \geq 0$ on $\mathbb{R}$, v - harmonic.

By the maximum principle, $v(z) > 0$ if $A \neq \emptyset$.

$\operatorname{hcap} A = \lim_{z \to \infty} -z(z - g_A(z)) = -\lim_{y \to \infty} iy \cdot i v(iy) = \lim_{y \to \infty} y\, v(iy) \geq 0$.

Observe that

$v(z) = \int_{\mathbb{R} \cup A} \operatorname{Im} z\, d\omega_z$ (because for $z \in \mathbb{R} \cup A$, $\operatorname{Im} g_A(z) = 0$). Thus

$\operatorname{hcap} A = \lim_{y \to \infty} y \int \operatorname{Im} z\, d\omega_{iy}$.

Assume now that A locally connected. Let $f_A := g_A^{-1}$ - extends to $f^*: \mathbb{C} \setminus I \to \mathbb{C} \setminus (A \cup \overline{A})$ (by reflection), where $I \subset \mathbb{R}$ - interval. Since A - l.c., $\exists$ an extension of f to I. Denote it by f_I.

Cauchy formula:

$$2\pi i f^*(w) = \underbrace{\int_{|z| = R} \frac{f^*(z)}{z - w} dz}_{I_R} + \underbrace{\int_I \frac{f_I(x) - \overline{f_I(x)}}{x - w} dx}_{\text{II}} \quad (R > |w|).$$

But $\lim_{R \to \infty} I_R = \lim_{R \to \infty} \int \frac{z}{z - w} dz = 2\pi i w$.

So $f^*(w) - w = \frac{1}{\pi} \int_I \frac{\operatorname{Im} f_I(x)}{x - w} dx$. Multiply by w. Take $w \to \infty$, to get

$$\operatorname{hcap}(A) = \frac{1}{\pi} \int_I \operatorname{Im} f_I(x)\, dx > 0.$$

Finally, an arbitrary A contains a locally connected non-empty subset ∎

Caratheodory convergence.

The natural notion of convergence in Complex Analysis.

Def Let Ω_n be a sequence of domains, $w_0 \in \cap \Omega_n$, $f_n: \mathbb{D} \to \Omega_n$ - conformal,

$f_n(0)=0$, $f_n'(0)>0$.
$\Omega_n \to \mathbb{C}$ if $f_n'(0)\to\infty$.
$\Omega_n \to \{w_0\}$ if $f_n'(0)\to 0$
$\Omega_n \to \Omega$ if $f_n \to f\ (:\mathbb{D}\to\Omega, f(0)=w_0)$ uniformly on compact subsets of $\mathbb{D}$.

Remark Same for $\mathbb{H}\to\mathbb{H}\setminus A$, $f(\infty)=\infty$ – by reflection.

Thm. $\Omega_n \to \Omega \iff$ 1) $\forall K\subset\Omega$ - compact $\exists N: n>N\ K\subset\Omega_n$.
2) $\forall U$ - open, $w_0\in U$, if $U\subset\Omega_n$ for infinitely many n, then $U\subset\Omega$.

Remark. Caratheodory limit is wrt. w_0!

Subordination.

in $\mathbb{D}$: $f, g:\mathbb{D}\to\mathbb{C}$, g-conformal.
We say that $f\prec g$ if $\exists\ \varphi:\mathbb{D}\to\mathbb{D}$, $\varphi(0)=0$; $f=g\circ\varphi$.
Equivalent: $f(0)=g(0)$, $f(\mathbb{D})\subset g(\mathbb{D})$ (take $\varphi:=g^{-1}\circ f$).
Properties: 1) $\{f(z): |z|<r\}\subset\{g(z): |z|<r\}$. ($|\varphi(z)|\le|z|$).
2) $|f'(0)|\le|g'(0)|$
3) $\max(1-|z|^2)|f'(z)|\le\max(1-|z|^2)|g'(z)|$.

In $\mathbb{H}$:
Let $f,g:\mathbb{H}\to\mathbb{C}$, g - conformal, $g(\infty)=0$, $\lim\limits_{z\to\infty} z g'(z)>0$.
$f\prec g$ if $\exists\ \varphi:\mathbb{H}\to\mathbb{H}$, $\varphi(\infty)=\infty$ $f=g\circ\varphi$, $\lim\limits_{z\to\infty} z\,\varphi(z)=1$.
Equivalently: $f(\mathbb{H})\subset g(\mathbb{H})$, $\lim\limits_{z\to\infty} z f(z)=\lim\limits_{z\to\infty} z g(z)$ (take $\varphi=g^{-1}\circ f$.
Observe that $\operatorname{Im}\varphi(z)\ge\operatorname{Im} z$. Thus
1) $\{f(z):\operatorname{Im} z>c\}\subset\{g(z):\operatorname{Im} z>c\}$.

2) $\max y|f'(x+iy)|\le\max y|g'(x+iy)|$. (Schwarz lemma for hyp. metric in $\mathbb{H}$).

Class $\mathcal{P}$ in $\mathbb{D}$: $p\in\mathcal{P}\iff p\prec\frac{1+z}{1-z}\iff \operatorname{Re}p>0$, $p(0)=1$.
Then 1) $\frac{1-|z|}{1+|z|}\le|p(z)|\le\frac{1+|z|}{1-|z|}$
2) $|p'(z)|\le\frac{2}{1-|z|^2}$.

Herglotz representation: $p\in\mathcal{P}\iff\exists\ \mu$ - probability on S^1:
$p(z)=\int\frac{\xi+z}{\xi-z}\,d\mu(\xi)$, $\operatorname{supp}\mu=\overline{\{\xi\in S^1:\varlimsup\limits_{r\to1-}\operatorname{Re}p(r\xi)>0\}}$.

Class $\mathcal{P}$: $p\in\mathcal{P}\iff p\prec\frac{-1}{z}\iff p:\mathbb{H}\to\mathbb{H}$, $\lim\limits_{z\to\infty} z p(z)=1$
$\mathcal{P}$ is compact class, since: $|p(x+iy)|\le\frac{1}{y}$..
$|p'(x+iy)|\le\frac{1}{y^2}$.

Herglotz representation: $p(z)=\int_{\mathbb{R}}\frac{d\mu(x)}{x-z}$.
("pf" Take Poisson representation $\operatorname{Im}p(\xi_1+i\xi_2)=\frac{1}{\pi}\int_{\mathbb{R}}\frac{\operatorname{Im}p(x)\,\xi_2}{(x-\xi_1)^2+\xi_2^2}\,dx$ and take conjugate).

Löwner chain (radial):

Def. $\{f_t\}$ - family of conformal mappings, $f_t:\mathbb{D}\to\mathbb{D}$, such that
1) $t_1<t_2\Rightarrow f_{t_1}\succ f_{t_2}$.
2) $f_t(z)$ is uniformly continuous in t on compact subsets of $\mathbb{D}$.
3) $f_0(z)=z$, $\lim\limits_{t\to\infty} f_t(z)=0$.
is called Löwner chain (non-normalized).

Geometric definition (equivalent!)
Family $\Omega_t\subset\mathbb{D}$, such that
1) $0\in\Omega_t\ \forall t$.
2) $\Omega_{t_1}\supset\Omega_{t_2}$. $t_1<t_2$.
3) $\Omega_0=\mathbb{D}$, $0\notin\operatorname{Int}\bigcap\limits_{t\ge0}\Omega_t$.
4) $t\to\Omega_t$ - Caratheodory continuous.

Examples. Slit domains, self-touching slits, growing hulls.
Same definition in $\mathbb{H}$ for Chordal chains.

Observe: $f_0'(0)=1$, $\lim_{t\to\infty} f_t'(0)=0$, $t\to f_t'(0)$ - continuous.
Thus can reparametrize: $f_t'(0)=e^{-t}$.

Def. Normalized (radial) Löwner Chain: $(1)-(3)+(4)$ $f_t'(0)=e^{-t}$.
Normalized (Chordal) Löwner Chain: LC + $\operatorname{hcap} A_t = 2t$.

Concentrate on radial, for now.
For $s<t$, define $\varphi_{s,t}(z) := f_s^{-1}\circ f_t(z) : \mathbb{D}\to\mathbb{D}$.
$\varphi_{s,t}'(0)=e^{s-t}$ $\quad =: \varphi(z,s,t)$.

Chain relation: $s\le t\le \tau$: $\varphi(z,s,\tau)=\varphi(\varphi(z,t,\tau),s,t)$.

Key observation:
$$p_{s,t}(z)=p(z,s,t) := \frac{1+e^{s-t}}{1-e^{s-t}}\,\frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)} \in \mathcal{P}.$$
So $p_{s,t}(z)=\int \frac{\xi+z}{\xi-z}\,d\mu_{s,t}(z)$, $\operatorname{supp}\mu_{s,t}\subset \overline{\{|\varphi_{s,t}(z)|\ne 1\}}$.

Now write:
$$\frac{f_t(z)-f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z))-f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z}\times$$
$$\frac{\varphi_{s,t}(z)-z}{t-s} = \frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z}\cdot\underbrace{\frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)}\,\frac{1+e^{s-t}}{1-e^{s-t}}}_{p_{s,t}(z)}\cdot\frac{(e^{s-t}-1)(z+\varphi_{s,t}(z))}{(e^{s-t}+1)(t-s)} \quad (*)$$

$$\frac{\partial f}{\partial t} = -f_t'(z)\,z\,p_t(z) \overset{\text{Herglotz}}{=} -f_t'(z)\,z\int\frac{\xi+z}{\xi-z}\,d\mu_t(\xi)$$
where $p_t(z)=\lim_{s\to t} p_{s,t}(z)$.

Derivation:
Notice that $|f_t(z)-f_s(z)| = \left|\int_{\varphi_{s,t}(z)}^{z} f_s'(\xi)\,d\xi\right| \le |z-\varphi_{s,t}(z)|\,C(|z|)e^{-s}$ (from Köbe).
Since $|p_{s,t}(z)|\le \frac{1+|z|}{1-|z|}$, we get
$$|\varphi_{s,t}(z)-z| \le \underbrace{|\varphi_{s,t}(z)+z|}_{\le 2|z|}\,\frac{1-e^{(s-t)}}{1+e^{s-t}}\,\frac{1+|z|}{1-|z|}\quad . \text{ So}$$
$|f_t(z)-f_s(z)|\le C_1(|z|)(e^{-t}-e^{-s})$.
Same proof gives
$$|\varphi_{t,u}(z)-\varphi_{s,u}(z)|\le C_2(|z|)\,(1-e^{s-t}),\ t>s.$$
Thus $t\to f_t(z)$ Lipschitz, so a.e. t differentiable in countably many points $\Rightarrow$ differentiable at every point (for analytic functions, convergence on dense set $\Rightarrow$ convergence everywhere).
Returning to $(*)$: $\exists \lim$ LHS - by Lipschitz, a.e. t.
Then $\lim_{s\to t} \frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z} = f_t'(z)$ - Carathéodory continuity.
$$\lim_{s\to t}\frac{(e^{s-t}-1)(z+\varphi_{s,t}(z))}{(e^{s-t}+1)\ldots} = -z \ (\text{again} \ldots)$$

$\dots (e^{\dots}+1)(t-s) \dots$, $\lim_{s\to t} \varphi_{s,t}(z) = z$).

So $\exists \lim_{s\to t} p_{s,t}(z) =: p_t(z)$.

We proved a half of

Thm (Löwner) (f_t) is normalized L.C. iff

1) f_t - holomorphic on $\mathbb{D}$, $t \mapsto f_t$ a.c. in t $\forall z$.

2) $\exists (p_t) \in \mathcal{P}$ - measurable in t, such that a.e. t $\forall z \in \mathbb{D}$.

$$\frac{\partial f_t}{\partial t} = -z \frac{\partial f_t(z)}{\partial z} p_t(z)$$

To prove the other direction, consider $\varphi_{s,t}(z)$.

They satisfy $\frac{d\varphi_{s,t}(z)}{ds} = -\varphi_{s,t}(z) p_s(\varphi_{s,t}(z))$, with $\varphi_{t,t}(z) = z$

So we consider

$w(s) := \varphi_{s,t}(z)$, $(s \le t)$, then $\frac{dw}{ds} = -w\, p_s(w)$, $w(t) = z$ $(**)$.

Thm (Löwner–Kufarev). The equation $(**)$ has unique solution $w^{t,z}(s)$ for $s \in [0,t]$ with $w^{t,z}(t) = z$. The map $\varphi_{s,t}(z) := w^{t,z}(s)$ is univalent, - . $\varphi_{0,t}(z)$ form Löwner chain (in t)

Löwner–Kufarev $\Rightarrow$ Löwner.

(This is the computation showing $\varphi_{s,t}$ sat ODE for L.C.)

$$\frac{\partial}{\partial s} f_s(w^{t,z}(s)) = f_s'(w)\frac{\partial w^{t,z}}{\partial s}(s) + \frac{\partial f}{\partial s}(w^{t,z}(s)) = 0, \text{ so}$$

$f_s(w^{t,z}(s))$ does not depend on s, i.e. $f_s(w^{t,z}(s)) = f_t(w^{t,z}(t)) = f_t(z)$.

Also $f_s(w^{t,z}(s)) = f_0(w^{t,z}(0)) = \varphi_{0,t}(z)$

Thus $f_t(z) = \varphi_{0,t}(z)$ – Löwner chain. ∎

Pf of Löwner–Kufarev (Picard–Lindelöf iteration).

Let $|z| = r$. Rewrite as an integral equation:

$$w(s) = z \exp\left(-\int_s^t p(w(\tau),\tau)\,d\tau\right).$$

Define: $w_0(s) = 0$

$$w_{n+1}(s) = z \exp\left(-\int^t p(w_n(\tau),\tau)\,d\tau\right).$$

Since $p \in \mathcal{P}$, $|p'(\xi,\tau)| \le \frac{2}{1-|\xi|^2}$. Thus

$$|w_{n+1}(s) - w_n(s)| \le \int_s^t |p(w_n(\tau),\tau) - p(w_{n-1}(\tau),\tau)|\,d\tau \le \frac{2}{(1-r)^2}\int_s^t |w_n(\tau) - w_{n-1}(\tau)|\,d\tau.$$

Since $\operatorname{Re} p > 0$, $|w_n(\tau)| \le r$. Thus,

$$|w_{n+1}(s) - w_n(s)| \le \frac{2^n (t-s)}{(1-r)^2 n!}, \text{ by induction on } n.$$

So $\exists \lim_{n\to\infty} w_n(s)$, uniform for $s \le t$ and in $|z| \le r$.

$w_n^{z,t}(s)$ is an analytic function of z, so is w.

By dominated convergence, $w(s) = z\exp\left(-\int_s^t p(w,\tau)\,d\tau\right)$, thus satisfying $(**)$.

Note now that $\frac{d|w|^2}{ds} = \frac{d w\bar{w}}{ds} = 2|w|^2 \operatorname{Re} p_s(w) > 0$, so $|w|$ is increasing in s.

Thus if w_1, w_2 - two solutions of $(*)$ if $w_1(t) = w_2(t) = z$, then $\forall s < t$ $|w_1(s) - w_2(s)| \le |w_1(t) - w_2(t)| = 0$, so $w_1 \equiv w_2$.

Define $\varphi(z,s,t) := w^{t,z}(s)$ $(s \le t)$. By uniqueness, we have the chain relation

$$\varphi(z,s,t) = \varphi(\varphi(z,\tau,t), s, \tau), \quad s \le \tau \le t.$$

Assume now that for some $T \le t$, $\varphi(z_1, T, t) = \varphi(z_2, T, t)$. Then for all $s \le T$, Chain relation give $\varphi(z_1, s, t) = \varphi(z_2, s, t)$.

Observe now that for any two solutions $w(t)$ and $v(t)$ of $(*)$, we have

$\left|\frac{d}{ds}(w(s) - v(s))\right| = |w\, p(w,s) - v\, p(v,s)| \le K|w - v|$, where K comes from the distortion bound for p. Thus, applying to $w(s) = \varphi(z, s, t)$, $v(s) = \varphi(z_2, s, t)$, we get $w \equiv v$ for all $s \ge T$, including $s = t$, so $z_1 = z_2$. Thus $\varphi(z, T, t)$ is conformal!

<u>Remark</u>. $g_t := f_t^{-1}$, then $\partial_t g_t = g_t(z)\, p(g_t(z), t)$.

$|\varphi'| \le K|\varphi| \Rightarrow$
$\psi = \varphi e^{-Kt}$
$\psi' = (\varphi' - K\varphi) e^{-Kt} \le 0$
if φ – real
Now, multiply by $e^{i\theta}$.