Loewner Evolution

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Some preliminaries:

1) Subordination.

Let $f, g \in A(\mathbb{D})$, g - conformal. We say that f is subordinate to g if $\exists\ \varphi: \mathbb{D} \to \mathbb{D}$ - analytic: $f = g \circ \varphi$.
$\varphi(0) = 0$

Equivalently: $f(\mathbb{D}) \subset g(\mathbb{D})$, $f(0) = g(0)$. Pf $\varphi = g^{-1} \circ f$.

Consequences of subordination:

a) $\{f(z) : |z| < r\} \subset \{g(z) : |z| < r\}$ (since $|\varphi(z)| \le |z|$).

b) $|f'(0)| \le |g'(0)|$ ($|\varphi'(0)| \le 1$).

c) $\max (1-|z|^2)|f'(z)| \le \max (1-|z|^2)|g'(z)|$ (By Schwarz: $\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \le \frac{1}{1-|z|^2}$).

2) Class $\mathcal{P}$

Def. $p \in \mathcal{P} \iff p \prec \frac{1+z}{1-z} \iff \operatorname{Re} p > 0$, $p(0) = 1$.
$p \in A(\mathbb{D})$

So 1) $\frac{1-|z|}{1+|z|} \le |p(z)| \le \frac{1+|z|}{1-|z|}$

2) $|p'(z)| \le \frac{2}{1-|z|^2}$

So $\mathcal{P}$ is locally bounded $\Rightarrow$ normal.

Herglotz representation: $p \in \mathcal{P} \Rightarrow \exists\ \mu$ - probability measure on $\mathbb{T}$:

$$p(z) = \int \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)$$

$\operatorname{supp} \mu = \operatorname{clos}\{\zeta \in \mathbb{T} : \lim_{r \to 1-} \operatorname{Re} p(r\zeta) \ne 0\}$.

Pf Write the Poisson representation for positive $\operatorname{Re} p$, take conjugate ∎

Classical Löwner Chain (or Radial Löwner Chain)

Def. (f_t) - collection of conformal mappings from $\mathbb{D}$, $t \in [0, \infty)$, such that

1) $t_1 < t_2 \Rightarrow f_{t_1} \succ f_{t_2}$

2) $f_t(z)$ is continuous in t, uniformly on compacts $z \in K \subset \mathbb{D}$.

3) $f_0(z) = z$, $\lim_{t \to \infty} f_t(z) = 0$

is called Löwner chain (non-normalized)

Equivalent geometric def (in terms of $\Omega_t := f_t(\mathbb{D})$).

1) $0 \in \Omega_t$ $\forall t$.

2) $t_1 < t_2 \Rightarrow \Omega_{t_1} \supset \Omega_{t_2}$. $K_t := \mathbb{D} \setminus \Omega_t$ - growing subsets, hulls.

3) $\Omega_0 = \mathbb{D}$, $0 \in \operatorname{Int} \bigcap_{t \to \infty} \Omega_t$.

4) $t \to \Omega_t$ is Carathéodory continuous wrt. 0.

Example. Slit domains, self-touching slit domains.

Observe $a(t) := |f_t'(0)|$ - continuous, $a(0) = 1$, $\lim_{t \to \infty} a_t = 0$, monotone (by 1))

Thus can reparametrize time, so that $f_t'(0) = e^{-t}$

Def. Normalized L.C. (or simply L.C.) 1)-3) + 4) $f'(0) = e^{-t}$.

For $s < t$, define $\varphi_{s,t}(z) = f_s^{-1} \circ f_t(z) : \mathbb{D} \to \mathbb{D}$ - conformal,

$\varphi_{s,t}'(0) = e^{s-t}$ Notation: $\varphi(z, s, t) := \varphi_{s,t}(z)$

Chain relation: $s \le t \le \tau$. Then

$$\varphi(z, s, \tau) = \varphi(\varphi(z, t, \tau), s, t).$$

$$(f_s^{-1} \circ f_\tau = (f_s^{-1} \circ f_t) \circ (f_t^{-1} \circ f_\tau)).$$

Key observation.

$$p_{s,t}(z) = p(z, s, t) := \frac{1 + e^{(s-t)}}{1 - e^{(s-t)}} \quad \frac{z - \varphi_{s,t}(z)}{z + \varphi_{s,t}(z)} \in \mathcal{P}. \quad s < t.$$

so $p_{s,t}(z) = \int \frac{\zeta + z}{\zeta - z}\, d\mu_{s,t}(\zeta)$, $\operatorname{supp} \mu_{s,t}(\zeta) = \operatorname{Cl}\{\zeta \in \mathbb{D} : \lim_{r \to 1-} |\varphi_{s,t}(r\zeta)| \ne 1\}$.

Key trick: Write

$$\frac{f_t(z) - f_s(z)}{t - s} = \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{t - s} = \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{\varphi_{s,t}(z) - z} \cdot \frac{\varphi_{s,t}(z) - z}{t - s} =$$

$$\frac{f_t(z)-f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z))-f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z}\cdot\frac{\varphi_{s,t}(z)-z}{t-s} =$$

$$\frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z}\cdot \underbrace{\frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)}\,\frac{1+e^{s-t}}{1-e^{s-t}}}_{p_{s,t}(z)}\;\frac{(e^{s-t}-1)(z+\varphi_{s,t}(z))}{(1+e^{s-t})(t-s)} \quad (*).$$

$p_{s,t}(z) = \int \frac{z+\zeta}{z-\zeta}\,d\mu_{s,t}$.

Formally, if we let $t\to s$, we get

$\frac{\partial f_t(z)}{\partial t} = -f_t'(z)\,z\int\frac{\zeta+z}{\zeta-z}\,d\mu_t$ — Löwner equation.

Now, let us prove it

Lemma. $(f_t(z))$ - L.C. Then

1) $|f_t(z)-f_s(z)| \le \frac{8|z|}{(1-|z|)^4}(e^{-t}-e^{-s})$

2) $|\varphi_{t,u}(z)-\varphi_{s,u}(z)| \le \frac{2|z|}{1-|z|^2}(1-e^{s-t})$ $t>s>u$.

Pf $|p_{s,t}(z)| \le \frac{1+|z|}{1-|z|}$, so

$|\varphi_{s,t}(z)-z| \le |\varphi_{s,t}(z)+z|\,\frac{1-e^{s-t}}{1+e^{s-t}}\,\frac{1+|z|}{1-|z|} \le 2|z|\frac{1+|z|}{1-|z|}(1-e^{s-t})$

Now $|f(z,t)-f(z,s)| = \left|\int_z^{\varphi_{s,t}(z)} f'(\zeta,s)\,d\zeta\right|$. Use distortion Thm to estimate $|f'(z,\zeta)|$. Same for φ.

So $t\to f_t(z)$ - Lipshitz $\Rightarrow$ a.e. differentiable for all z

$(*)$ again. $\lim_{t\to s}$ LHS exists a.e. in t, by Lipshitz (do it for dense set, use uniform continuity on compacts.

$\lim_{t\to s}\frac{f_s(\varphi_{s,t}(z))-f_s(z)}{\varphi_{s,t}(z)-z} \to f_s'(z)$, by Caratheodory convergence ($\varphi_{s,t}(z)\to z$ unif. on compacts).

$\lim_{t\to s}\frac{e^{(s-t)}-1}{1+e^{s-t}}\,\frac{(z+\varphi_{s,t}(z))}{t-s} = -z$, again since $\varphi_{s,t}(z)\to z$.

So $\exists \lim_{t\to s} p_{s,t}(z) =: p_s(z)$ a.e. s.

Same reasoning for $t<s$ - again $\varphi_{t,s}(z)\to z$ as $t\to s$.

So we just proved half of Löwner Thm.

Thm (f_t) is a normalized L.C. iff

1) $\forall t$ $f_t\in A(\mathbb{D})$, $t\to f_t(z)$ - absolutely continuous $\forall z\in\mathbb{D}$.

2) $\exists (p_t)$ - measurable in t family of functions from $\mathcal{P}$, such that a.e. t, $\forall z\in\mathbb{D}$, $\frac{\partial f_t}{\partial t} = -z\frac{\partial f_t(z)}{\partial z}p_t(z)$.

Let us now prove the other direction.

First, let us consider $\varphi_{s,t}(z)$.

Observe: $0 = \frac{\partial}{\partial s}f_t(z) = \frac{\partial}{\partial s}f(\varphi(z,s,t),s) = \frac{\partial}{\partial s}f(\varphi,s) + f'(\varphi,s)\cdot\frac{\partial}{\partial s}\varphi(z,s,t)$

On the other hand, by Löwner at $\varphi(z,s,t)$

$\frac{\partial}{\partial s}f(\varphi,s) + \varphi p(\varphi,s)f'(\varphi,s) = 0$.

Then, since $f'(\varphi,s)\ne 0$, $\boxed{\frac{\partial}{\partial s}\varphi(z,s,t) = \varphi(z,s,t)\,p(\varphi(z,s,t),s).}$

Remark $\frac{\partial}{\partial s}f_s^{-1}\circ f_t = f_s^{-1}\circ f_t\, p_s(f_s^{-1}\circ f_t)$ $g_s := f_s^{-1}$-inverse $\frac{\partial}{\partial s}g_s = g_s\,p_s(g_s)$

So if $w(s) := \varphi(z,s,t)$, we have $\frac{dw}{ds} = w\,p_s(w)$, defined for $s\le t$.

with b.c. $w(t) = \varphi(z,t,t) = z$ $(**)$.

Let us study $(**)$.

Thm. (Löwner - Kufarev).

Equation $\frac{dw}{ds} = w\,p_s(w)$ a.e. s has unique solution $w^{t,z}(s)$ for $s\in[0,t]$ with $w^{t,z}(t)=z$. The map $z\to\varphi_{s,t}(z) := w^{t,z}(s)$ is univalent.

The maps $(\varphi_{0,t}(z))$ form normalized Löwner chain.

How Löwner–Kufarev implies other direction of Löwner Thm.

Let (f_s) satisfy 1) and 2) of Löwner. All a.e., but fine by absolute continuity.

Then $\frac{\partial}{\partial s} f_s(w^{t,z}(s)) \overset{\text{chain rule}}{=} f_s'(w)\frac{\partial w^{t,z}(s)}{\partial s} + \frac{\partial f_s(w)}{\partial s} \overset{\text{Löwner}}{=} 0$

So $f_s(w^{t,z}(s))$ is constant in s, equal to $f_t(z)$.

So $f_t(z) = f_s(\varphi_{s,t}(z))$. In particular, $f_t(z) = f_0(\varphi_{0,t}(z)) = \varphi_{0,t}(z)$ – L.C.

Proof of Löwner–Kufarev.

Use Pickard–Lindelöf iteration.

For now, fix z, $r := |z|$.

Rewrite as integral equation:

$w(s) := z\exp\left(-\int_s^t p(w(\tau),\tau)\,d\tau\right)$

Let $w_0(s) := 0$

$w_{n+1}(s) := z\exp\left(\int_s^t p(w_n(\tau),\tau)\,d\tau\right)$. Observe: $|w_n(s)| \le |z|$.

Note: $p \in \mathcal{C}$, so $|p'(\zeta,\tau)| \le \frac{2}{(1-|\zeta|)^2}$; also, for $\operatorname{Re} a < 0$, $\operatorname{Re} b < 0$: $|e^a - e^b| \le |a-b|$

so $|w_{n+1}(s) - w_n(s)| \le \int_s^t |p(w_n(\tau),\tau) - p(w_{n-1}(\tau),\tau)|\,d\tau \le \frac{2}{(1-r)^2}\int_s^t |w_n(\tau) - w_{n-1}(\tau)|\,d\tau$

Thus, by induction, $|w_{n+1}(s) - w_n(s)| \le \frac{2^n}{(1-r)^{2n}}\frac{(t-s)^n}{n!}$

Thus $\lim_{n\to\infty} w_n(s)$ exists, uniformly on compacts.

Also, for each fixed $s \le t$, $z \to w_n(s)$ is analytic in z, so is w.

By Lebesgue bounded convergence

$w(s) = z\exp\left(-\int_s^t p(w,\tau)\,d\tau\right)$. Thus satisfies $(**)$.

observe: $\frac{d|w|^2}{ds} = 2|w|^2 \operatorname{Re} p_s(w)$ so $|w|$ is increasing in s.

Now note that $\left|\frac{d}{ds}(w(s) - v(s))\right| = |w\,p(w,s) - v\,p(v,s)| \le k(r)\,|w - v|$, where k comes from distortion for $\mathbb{D}$. Thus if $w(t) = v(t)$ then for all $s < t$ $w(s) = v(s)$. Thus, we have the uniqueness, by Gronwall.

By uniqueness, we have the chain relation $\varphi(z,s,T) = \varphi(\varphi(z,t,T), s, t)$, $s \le t \le T$.

(both satisfy equation in s, and for $s = t$ LHS $= \varphi(z,t,T)$ = RHS

$\varphi(\varphi(z,t,T),t,t) = \varphi(z,t,T)$).

By the same uniqueness, $\varphi(z_1,s,t) = \varphi(z_2,s,t) \Rightarrow \varphi(z_1,t,t) = \varphi(z_2,t,t) \Rightarrow z_1 = z_2$, so φ is univalent.

Observe that by chain relation with $s = 0$

$\varphi(z,0,T) = \varphi(\varphi(z,t,T),0,t)$. So if $f_t := \varphi(z,0,t)$, we get

$f_T = f_t(\varphi(z,t,T))$ for $T > t$. Thus

$0 = \frac{\partial}{\partial t} f_T = \frac{\partial}{\partial t} f_t(\varphi(z,t,T)) = \frac{\partial}{\partial t} f_t(\varphi(z,t,T)) + f_t'(\varphi(z,t,T))\frac{\partial}{\partial t}\varphi(z,t,T) = \frac{\partial}{\partial t} f_t(\varphi(z,t,T)) +$

$f_t'(\varphi(z,t,T))\,\varphi(z,t,T)\,P(\varphi(z,t,T),t)$. Take $T = t$ to get

$\frac{\partial}{\partial t} f_t(z) = -f_t'(z)\,z\,P_t(z)$

Consider a very special case of slit domains.

Let $\gamma(t) : [0,\infty] \to \mathbb{D}$, $\gamma(0) \in \partial\mathbb{D}$, $\gamma(\infty) = 0$, $\gamma(0,\infty) \subset \mathbb{D}\setminus\{0\}$, simple (not self-touching). Normalize so that $f_t : \mathbb{D} \to \mathbb{D}\setminus\gamma[0,t)$ has $f_t'(0) = e^{-t}$.

Then $\Omega_t :=$ component of 0 of $\mathbb{D}\setminus\gamma[0,t]$ of normalized L.C.

f_t extends continuously to $\overline{\mathbb{D}}$ (by Carathéodory!) Let $\lambda(t) := f_t^{-1}(\gamma(t))$. As before $g_t := f_t^{-1}$.

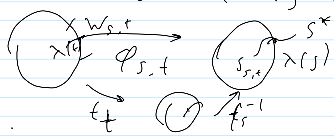
Theorem. $\frac{d}{dt} g_t(z) = g_t(z)\frac{\lambda(t) + g_t(z)}{\lambda(t) - g_t(z)}$ $\qquad \frac{\partial}{\partial t} f_t(z) = -z f_t'(z)\frac{\lambda(t)+z}{\lambda(t)-z}$

Remark. This means that $\mu_t = \delta_{\lambda(t)}$, where $\lambda(t)$ changes continuously.

$\lambda(t)$ is called driving function of γ.

Pf. Remember that $p(z,s,t) = \int \frac{\xi+z}{\xi-z}\, d\mu_{s,t}(\xi)$, where $\mu_{s,t}$ is supported on the set where $|\varphi_{s,t}(z)| < 1$, i.e. the set $f_t^{-1} \circ f_s(\gamma) = f_t^{-1}(\gamma[s,t]) = g_t(\gamma[s,t]) =: W(s,t)$ - notation.

Let $S_{s,t} := g_s(\gamma[s,t]) = \varphi_{s,t}(W(s,t))$



Use Schwarz reflection to expand $\varphi_{s,t}$ to map $\hat{\mathbb{C}} \setminus W(s,t) \to \hat{\mathbb{C}} \setminus (S_{s,t} \cup S^*_{s,t})$.

By Caratheodory, since $\lim_{s \to t} \varphi_{s,t}(z) = z$, $\lim_{s \to t} \operatorname{diam} S_{s,t} = 0$

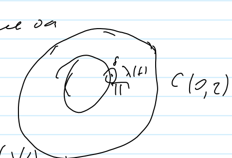
also $\lim_{t \to s} \varphi_{s,t}(z) = z$, so $\lim_{t \to s} \operatorname{diam} S_{s,t} = 0$.

Thus, by Lavrentiev estimate, $\lim_{\substack{t \to s \\ s \to t}} \operatorname{diam} W_{s,t} = 0$. Thus when $t \downarrow s$, $S_{s,t}$ approaches $\lambda(s)$.

Let us show that $S_{s,t}$ approaches $\lambda(t)$ when $s \uparrow t$.

Know: $\forall \varepsilon > 0$ $W_{s,t} \subset B(\lambda(t), \varepsilon)$ if s is close to t.

$\varphi_{s,t}(z) \xrightarrow[s \to t]{} z$ uniformly on compacts in $\mathbb{D}$. By reflection, same is true on compacts in $\mathbb{C} \setminus \mathbb{T}$. K contour $C(0,2) \to C(\lambda(t), \delta)$, $\delta < \varepsilon$.



Take Cauchy for $\varphi_{s,t}$ on $B(\lambda(t), \delta)$ (works when $S_{s,t} \subset B(\lambda(t), \delta)$).

Then $\varphi_{s,t}(z) \to z$ pointwise, so $\varphi_{s,t}(z) \to z$ uniformly on $C(\lambda(t), \varepsilon)$.

Thus, for s close to t, $\varphi_{s,t}(B(\lambda(t), \varepsilon)) \subset B(\lambda(t), 2\varepsilon)$, so $S_{s,t} \subset B(\lambda(t), 2\varepsilon)$

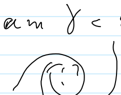
Thus $\forall \varepsilon$ $|\lambda(s) - \lambda(t)| < 2\varepsilon$ if s close enough to t.

Thus $\lambda(t)$ is continuous in t. $\mu_{s,t}$ is supported on $W_{s,t}$. As $s \to t$, $\mu_{s,t} \to \delta_t$ (since it is the only probability measure supported on $\delta_{\lambda(t)}$). Thus $\lim_{s \to t} p_{s,t}(z) = \frac{\lambda(t)+z}{\lambda(t)-z}$. Moreover, $\frac{dg_t}{dt}$ exists for all t (if you review the equation).

What can be generated by continuous functions? Unfortunately, not only curves.

Thm. (Pommerenke) The L.C. (Ω_t) has a continuous driving function $\lambda(t)$ iff $\forall T > 0$, $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall t \leq T$ $\exists$ crosscut γ in Ω_t separating 0 from $\Omega_t \setminus \Omega_{t+\delta}$, $\operatorname{diam} \gamma < \varepsilon$.

Example (bad) Spiral.



Pf. If you examine the proof of previous Thm, the existence of $\lambda(t)$ is proven ~~the same way~~.

Other direction:

Fix $\eta > 0$, $\delta < \eta^2/16$: $|t-s| \leq \delta \Rightarrow |\lambda(t) - \lambda(s)| < \frac{\eta}{4}$.

Claim 1. $|z| < 1$, $|z - \lambda(s)| > \eta$. Then $u(t) := |\lambda(s) - \varphi_{s,t}(z)| > \eta/2$ and

$|z - \varphi_{s,t}(z)| < \frac{\eta}{2}$ for $s \leq t \leq s+\delta$.

Meaning: z that was far away from the tip image, should stay far away for time δ.

Pf. Let $t_1 = \inf\{t : u(t) = \eta/2\}$. For $s \leq t \leq t_1$, $u(t) \geq \frac{\eta}{2}$. So

$|\lambda(t) - \varphi_{s,t}(z)| \geq |\lambda(s) - \varphi_{s,t}(z)| - |\lambda(t) - \lambda(s)| \geq \frac{\eta}{4}$ if $t_1 - s < \delta$

so $\frac{d}{dt} u(t) = \frac{\partial}{\partial t} |\lambda(s) - \varphi_{s,t}(z)| \geq -|\varphi_{s,t}(z)| \left| \frac{\lambda(t) + \varphi_{s,t}(z)}{\lambda(t) - \varphi_{s,t}(z)} \right| \geq \frac{8}{\eta}$.

so $u(t_1) - u(s) \geq -\frac{8}{\eta}(t_1 - s) \geq -\frac{8\delta}{\eta} > -\frac{\eta}{2}$ - contradiction!

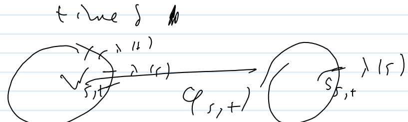
Claim 2. Let $|z - \lambda(s)| > \eta$. Then $|\varphi_{s,t}(z)| > |z|^2$ when $s \leq t \leq s+\delta$.

Pf. Let $t_2 = \inf\{t : |\varphi_{s,t}(z)| = |z|^2\}$.

Then

$$\partial_t \log|\varphi_{s,t}(z)| = \operatorname{Re} \left| \frac{1}{\varphi} \frac{\partial \varphi}{\partial t} \right| = -\operatorname{Re}\left(\frac{\lambda(t) + \varphi}{\lambda(t) - \varphi} \right) = \frac{|\varphi|^2 - 1}{|\lambda(t) - \varphi|^2}.$$

As before can see that $\log \varphi$ cannot go from $\log|z|$ to $2\log|z|$ in time δ.



$S_{s,t} = \mathbb{D} \setminus \varphi_{s,t}(\mathbb{D}) = f_s^{-1}(\Omega_s \setminus \Omega_t)$, as before

By Claims 1 and 2, if $s \leq t \leq s+\delta$, then $S_{s,t} \subset \{z \in \mathbb{D} : |z - \lambda(s)| < 2\eta\}$.

Indeed, $w \in S_{s,t} \Rightarrow w = \varphi_{s,t}(z)$, $|w|^2 < |z|^2$, so, by claim 2, $|z - \lambda(s)| \leq \eta$

By Claims 1 and 2, if $s \le t \le s+\delta$, then $S_{s,t} \subset \{z \in \mathbb{D} : |z-\lambda(s)| < 2\eta\}$.

Indeed, $w \in S_{s,t} \Rightarrow w = \varphi_{s,t}(z)$, $|w|^2 < |z|^2$, so, by claim 2, $|z-\lambda(s)| \le \eta$.
If $D_\eta = \{z : |z-\lambda(s)| > \eta\}$, $\varphi_{s,t}(D_\eta) \cap S_{s,t} = \emptyset$. On the other hand,
$\varphi_{s,t}(D_\eta) \supset \{w : |w-\lambda(t)| > 2\eta\}$, since $|\lambda(s)-\lambda(t)| < \eta/4$ and Claim 1.

So, by Wolf Lemma, we can find a ~~short~~ crosscut in $f_s(\mathbb{D})$ separating $\gamma[s,t] = f_s(S_{s,t})$ from 0 ∎

Remark again that $\gamma(t)$ can be self-touching or even space-filling.

Def. L.C. is generated by a curve γ if
Ω_t = component of 0 of $\mathbb{D} \setminus \gamma[0,t]$

Thm. TFAE for L.C. with a continuous driving function $\lambda(t)$:

1) L.C. is generated by a curve.
2) K_t is locally connected $\forall t$.
3) $\lim_{r\to 1} f_t(r\lambda(t)) =: \gamma(t)$ $\exists$ and continuous in t.

Pf. 1) $\Rightarrow$ 2) – obvious. 2) $\Rightarrow$ 3) – $\exists$ – obvious; continuity – Pommerenke.

3) $\Rightarrow$ 1) Note: $\gamma(t) \notin \Omega_t \Rightarrow \gamma[0,t] \subset K_t$ $\forall t$.
Enough to prove: $\partial\Omega_t \subset \gamma[0,t] \cup \partial\mathbb{D}$.
Let $z \in \partial\Omega_t \setminus \partial\mathbb{D}$, $\varepsilon > 0$, and $t' = \sup\{s : K_s \cap \overline{B(z,\varepsilon)} = \emptyset\}$. $t' \le t$.
Take $p \in B(z,\varepsilon) \cap \Omega_{t'}$, $p' \in K_{t'} \cap \overline{B(z,\varepsilon)}$. And let p'' be the first point of segment from p to p' which is $K_{t'}$. Then $L = [p,p'')$ – semi-crosscut. So $g_{t'}(L)$ terminates at some $x \in \partial\mathbb{D}$. If $x \ne \lambda(t')$, then for $s < t'$, $|s-t'|$ – small, we have $x \ne \lambda(s)$, and $g_s(L)$ terminates at $\varphi_{st'}(x) \in \partial\mathbb{D}$! Thus $p'' \in K_s$ – contradiction! So $p'' = \gamma_{t'}$.
Take $\varepsilon \to 0$. Then $z = \lim_{\varepsilon\to 0} \gamma_{t'_\varepsilon} \in \gamma[0,t]$ ∎

Best deterministic conditions so far:

Thm (Lind). 1) If $\lambda(t) \in \text{Höl}_{1/2}$, $|\lambda(t)-\lambda(s)| \le C|s-t|^{1/2}$, $C<4$, then K_t is generated by a curve.
2) $\exists$ $\lambda(t) \in \text{Höl}_{1/2}$, $|\lambda(t)-\lambda(s)| \le 4|s-t|^{1/2}$, such that K_t is not generated by a curve.