Martingales.

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Filtrations, adapted processes, stopping times.

Def. $(\Omega, \mathcal{F})$ - measurable space. Filtration is an increasing family of sub-σ-algebras of $\mathcal{F}$. A measurable space with a filtration is called a filtered space. $(\mathcal{F}_t)_{t\in I}$, I - some index set. For us, $I = \mathbb{N}$ or $I = \mathbb{R}_+$.

Examples: 1) A b-adic filtration of $([0,1), \mathcal{B})$: $\mathcal{F}_n$ - generated by the b-adic intervals of n-th generation

2) **Def.** A process is called adapted to a filtration if X_t is $\mathcal{F}_t$ measurable $\forall t$.

Each process has a natural filtration - the smallest filtration makes X_t $\mathcal{F}$-measurable

Def. A stopping time T wrt filtration $(\mathcal{F}_t)_{t\in I}$ is a random function $T: \Omega \to I$ such that $\forall t \in I \quad \{\omega : T(\omega) \le t\} \in \mathcal{F}_t$.

$$\mathcal{F}_T := \{A \in \mathcal{F}_\infty : A \cap \{T \le t\} \in \mathcal{F}_t \ \forall t\}.$$

Examples: 1) $I = \mathbb{N}$, $T = \min\{n : X_n \in A\}$ - hitting time. $\left(\{T = n\} = \{\omega : X_n \in A\} \cap \bigcap_{k<n} \{\omega : X_k(\omega) \notin A\}\right)$

2) $I = \mathbb{R}$, X_t a.s. continuous as a function of t, then $T = \inf\{t : X_t \in A\}$ - stopping time.

$$\left(\{\omega : T(\omega) \le t\} = \bigcap_n \bigcup_{\substack{q \le t \\ q \in \mathbb{Q}}} \left\{ \operatorname{dist}(X_q(\omega), A) < \frac{1}{n} \right\}\right)$$

Def. Submartingale wrt $(\mathcal{F}_t)_{t\in I}$.

X_t - $(\mathcal{F}_t)$-adapted.

submartingale if $\forall s < t, s, t \in I$:

$X_s \le E(X_t | \mathcal{F}_s)$.

supermartingale $(\ge)$ - $-X_t$ - submartingale

martingale = sub- + super martingale.

Example. Dyadic martingales:

Main example f - $\mathcal{F}_\infty$ - measurable. Then

$f_s := E(f | \mathcal{F}_s)$ - martingale.

When does it occur?

To answer this question, we first need

Lemma (Maximal inequality). Let $(X_n)_{n=1}^N$ be a discrete submartingale

Then $\forall \lambda > 0 \qquad \lambda P(X^* \ge \lambda) \le \int_{X^* \ge \lambda} |X_N| \, dP.$

Here $X^*(\omega) := \sup_{t\in I} X_t(\omega)$ - maximal function. Note that X_n - submartingale $\Rightarrow$ $|X_n|$ - submartingale, and thus $t \to \int |X_t| \, dP$ is increasing.

Pf. Let $T(\omega) := \begin{cases} \min\{n : X_n(\omega) \ge \lambda\}, & X^* \ge \lambda \\ N, & X^* < \lambda \end{cases}$

$t \in I$ ... that: $|X_t|$ - submartingale, and thus $t \to \int |X_t|\, dP$ is increasing.

Pf. Let $T(\omega) := \begin{cases} \min\{n : X_n(\omega) \geq \lambda\}, & X^* \geq \lambda \\ N, & X^* < \lambda \end{cases}$

Then $E(|X_N|) = \sum E(|X_N| \cdot \chi_{\{T=n\}} \chi_{X^* \geq \lambda}) + E(|X_N| \chi_{X^* < \lambda}) \geq$
$\sum_{n=1} E(|X_n| \chi_{\{T=n\}} \chi_{X^* \geq \lambda}) + \int_{X^* < \lambda} |X_N|\, dP \geq$
$\lambda \sum P(\chi_{\{T=n\}} \chi_{X^* \geq \lambda}) + \int_{X^* < \lambda} |X_N|\, dP = \lambda P(X^* \geq \lambda) + \int_{X^* < \lambda} |X_N|\, dP$, and we get the statement. ∎

Corollary. $P(X^* > \lambda) \leq \dfrac{\sup E(|X_t|)}{\lambda}$ for any submartingale (X_t) which is either discrete or continuous.

Pf. First, for discrete. Let $X_N^* := \max_{n \leq N} X_n$, then $X^* = \lim_{N \to \infty} X_N^*$, increasing, so $P(X^* > \lambda) = \sup P(X_N^* > \lambda) \leq \sup \dfrac{E(|X_N|)}{\lambda}$.

For continuous, consider $Y_n^{N,t} := X_{t \wedge 2^{-N}}$. It is a martingale with respect to $\mathcal{L}_n := \mathcal{F}_{t \wedge 2^{-n}}$. By continuity, $X^* = \lim_{t \to \infty} \lim_{N \to \infty} (Y^{N,t})^*$, and we can use the discrete inequality. ∎

Corollary. Let (X_t) be a submartingale (with continuous trajectories in continuous case). Then $E(X^{*p}) \leq \left(\frac{p}{p-1}\right)^p \sup_t |X_t|^p$.

Pf. As before, enough to prove for finite index set.
Let μ be the law of X^* ($\mu([a,b)) = P(a \leq X^* < b)$). Then
$E(|X^*|^p) = \int \lambda^p d\mu = \int_0^\infty p \lambda^{p-1} P(X^* \geq \lambda)\, d\lambda \leq \int_0^\infty p \lambda^{p-1} \left(\frac{1}{\lambda} \int_{|X| > \lambda} |X_N|\, dP\right) d\lambda =$

$p E\left(|X_N| \int_0^{X^*} \lambda^{p-2}\, d\lambda\right) = \frac{p}{p-1} E(|X_N|\, |X^*|^{p-1}) \underset{\text{Younge}}{\leq} \frac{p}{p-1} E(|X_N|^p)^{1/p}\, E(|X^*|^p)^{\frac{p-1}{p}}$ ∎

Corollary. (X_t) - martingale (continuous in continuous case), and $\sup_{t \in I} E(|X_t|) < \infty$.
Then $\overline{\lim} |X_t| < \infty$ a.s.

Pf. $P(\overline{\lim} |X_t| > \frac{1}{\varepsilon}) \leq P(X^* \geq \frac{1}{\varepsilon}) \leq \varepsilon \sup E(|X_t|) < \infty$ ∎

Remark. With a bit more work, one can prove that in this situation, $\lim_{t \to \infty} X_t$ exists a.s. Will prove a particular case.

Thm (Martingale convergence Thm).
Let $p > 1$, (X_n) - discrete martingale, $\sup_n E(|X_n|^p) < \infty$. Then $\exists X_\infty$:
1) $E(|X_\infty|^p) < \infty$, X_∞ is $\mathcal{F}_\infty$-measurable.
2) $X_n \to X_\infty$ a.s. and in L^p ($p < \infty$).
3) $X_n = E(X_\infty | \mathcal{F}_n)$.

Remark. Same is true in continuous case, by discrete approximation.

Pf. Enough for $p < \infty$ (for $p = \infty$ only need $\|X_\infty\|_\infty < \infty$, which follows from a.s. convergence).
Take a subsequence X_{n_k} weakly converging in L^q ($\frac{1}{q} + \frac{1}{p} = 1$), $X := \lim_{k \to \infty} X_{n_k}$ (weak).
For $S \in \mathcal{F}_n$, $\chi_S \in L^q$, so that:
$\int \chi_S X_n\, dP = \lim_{j \to \infty} \int X_{n_j} \chi_S\, dP = \int \chi_S X\, dP$, so $\int_S X_n\, dP = \int_S X\, dP$, i.e. $X_n = E(X_\infty | \mathcal{F}_n)$.
So X_∞ is unique on $\mathcal{F}_\infty$, and thus $X_n \xrightarrow{w} X_\infty$ in L^p.
Let $\Omega_X(\omega) := \overline{\lim} X_n(\omega) - \underline{\lim} X_n(\omega)$ - a.s. finite.
So $P(\Omega_X(\omega) > \varepsilon) \leq P(X^* > \varepsilon) + P((-X)^* > \varepsilon) \leq \frac{2}{\varepsilon} \int |X_\infty|\, dP$.
Let now $\mathcal{D} := \{g \in L^p : g \text{ is } \mathcal{F}_n\text{-measurable for some } n\}$. $\mathcal{D}$ is dense in L^p.

so $\forall \varepsilon>0\ \exists g_\varepsilon \in \mathcal{D}$: $E(|X_\infty - g_\varepsilon|^p) < \varepsilon^2$. Then, $P(\Omega_X > \varepsilon) = P(\Omega_{X-g} > \varepsilon) \le \frac{2}{\varepsilon}\cdot\varepsilon^2 = 2\varepsilon$.
So $\Omega_X = 0$ a.s., thus $\lim_{n\to\infty} X_n$ exists a.s.
Also, $E(|X_n - X_\infty|^p)^{1/p} \le E(E(X - g_\varepsilon|\mathcal{F}_n)^p)^{1/p} + E((g_\varepsilon)_n - g)^p)^{1/p} = \underset{\to 0 \text{ for large } n}{}$
$E(|X - g_\varepsilon|^p)^{1/p} < \varepsilon\ \forall \varepsilon$ for large n.
So $\lim X_n = X_\infty$ a.s. and in L^p ∎

In fact, assuming a.s. convergence (technical!), one can prove more.

Def. Let (X_t) be $(\mathcal{F}_t)$-adapted process. Then (X_t) is called uniformly integrable (u.i.) if $\forall \varepsilon>0\ \exists \delta>0: \forall t\ \forall E \in \mathcal{F}_t: P(E)<\delta \Rightarrow \int_E |X_t|\,dP < \varepsilon$.

Example. $\sup_{p>1} |X_t|^p < \infty \Rightarrow (X_t)$ u.i.

Pf. $\int_E |X_t|\,dP \le \left(\int_E |X_t|^p dP\right)^{1/p} \left(\int_E \chi_E^q\right)^{1/q} \le \|X_t\|_p^p\, |E|^{1/q}$ ∎

Thm. Let (X_t) be a discrete or continuous martingale.
TFAE
1) $\exists \lim_{t\to\infty} X_t = X_\infty$ in L^1.
2) $X_t = E(X_\infty|\mathcal{F}_t)$ for some X_∞.
3) X_t is u.i.

Pf. 1) ⇒ 2) L^1-conv implies for $\forall E \in \mathcal{F}_t$, $\int_E X_t\,dP = \lim_{s\to\infty} \int_E X_s\,dP = \int_E X_\infty\,dP$.
2) ⇒ 3) Take δ for ε that works for X_∞.
3) ⇒ 1) $X_t \to X_\infty$ a.s. (have to assume!) + u.i. ⇒ $X_t \to X_\infty$ in L^1, since
$$\int |X_t - X_\infty|\,dP \le \underbrace{\varepsilon P(|X_t - X_\infty| \le \varepsilon)}_{\le \varepsilon} + \underbrace{\int_{|X_n - X_\infty|>\varepsilon} |X_n - X_\infty|\,dP}_{\to 0 \text{ by u.i.}}$$ ∎

Now let us concentrate on discrete martingales.

Thm (Discrete Itô integration). Let H_n be $\mathcal{F}_{n-1}$-measurable (predictable process), $H_n \ge 0$, bounded. Let X_n be a martingale. Let $Y_0 := X_0$, $Y_n := Y_{n-1} + H_n(X_n - X_{n-1})$.
Then Y_n is also $\mathcal{F}_n$-martingale. Notation: $Y_n = (H\cdot X)_n$.

Pf. $E(Y_n|\mathcal{F}_{n-1}) = E(Y_{n-1} + H_n(X_n - X_{n-1})|\mathcal{F}_{n-1}) = Y_{n-1} + H_n E(X_n - X_{n-1}|\mathcal{F}_{n-1}) = Y_{n-1}$ ∎

Corollary. If T – stopping time, $X_n^T := X_{\min(n,T)}$. Then X_n^T is a martingale (called stopped martingale).

Pf. Take $H_n = \chi_{T\ge n} = 1 - \chi_{T\le n-1}$ – $\mathcal{F}_{n-1}$ measurable. ∎

Thm (Discrete Optional Stopping Time).
If $S \le T$ – two bounded stopping times (i.e. $\exists M \in \mathbb{N}: M \ge T \ge S$), then
$X_S = E(X_T|\mathcal{F}_S)$.

Pf. Let $H_n := \chi_{T\ge n} - \chi_{S\ge n}$.
$(H\cdot X)_n - X_0 = X_T - X_S$, for $n > M$, so $E(X_T - X_S) = E(X_0 - X_0) = 0$.
Take now $B \in \mathcal{F}_S$, consider $S^B := S\chi_B + M\chi_{B^c}$, $T^B := T\chi_B + M\chi_{B^c}$,
$S^B \le T^B$, so $E(S^B) = E(T^B) \Rightarrow E(S\chi_B) = E(T\chi_B) \Rightarrow E(T|\mathcal{F}_S) = S$ ∎

Def. Quadratic variation of a martingale (X_n) is
$$S_n := \sum_{j=1}^n E((X_j - X_{j-1})^2|\mathcal{F}_{j-1}) = S_{n-1} + E(X_n^2 - 2X_nX_{n-1} + X_{n-1}^2|\mathcal{F}_{n-1}) =$$
$S_{n-1} + E(X_n^2|\mathcal{F}_{n-1}) - X_{n-1}^2 -$

$S_n := \sum_{j=1} E((X_j - X_{j-1})^2 | \mathcal{F}_{j-1}) = S_{n-1} + E(X_n^2 - 2X_nX_{n-1} + X_{n-1}^2 | \mathcal{F}_{n-1}) =$
$S_{n-1} + E(X_n^2 | \mathcal{F}_{n-1}) - X_{n-1}^2$

Note that S_n is predictable, and $X_n^2 - S_n$ is a martingale.

$(E(X_n^2 - S_n | \mathcal{F}_{n-1}) = E(X_n^2 | \mathcal{F}_{n-1}) + X_{n-1}^2 - S_{n-1} - E(X_n^2 | \mathcal{F}_n).$

Note that for dyadic martingale,

$E((X_n - X_{n-1})^2 | \mathcal{F}_{n-1}) = (X_n - X_{n-1})^2$, (increase on the left = decrease on the right), so

$S_n = \sum (X_n - X_{n-1})^2.$

Thm (Levy) Let X_n be a martingale, $A = \{\omega : S = \lim_{n\to\infty} S_n < \infty\}$. Then a.s. on A $\exists \lim_{n\to\infty} X_n$.

Pf Let $T := \inf\{n-1 : S_n \geq M\}$. Because S_n is predictable, T is a stopping time, X_n^T is a martingale. $S_n(X^T) \leq M$ (it is $= S_n$ for $n \leq T$, and then stays the same).

Observe that $E(X_n^2) = E(S_n) + E(X_0^2)$, since $X_n^2 - S_n$ is a martingale.

Thus $\sup \|X_n^T\|_2^2 < \infty$, so $\forall M$ $\exists \lim_{n\to\infty} X_n^T = \lim_{n\to\infty} X_n$ if $T < M$, by Martingale Convergence Thm. Thus a.s. on $\{S < \infty\}$ $\exists \lim_{n\to\infty} X_n$. ∎

What happens on $\{S = \infty\}$?

Thm. Let X_n be a real dyadic martingale. Then

$$\overline{\lim} \frac{X_n}{\sqrt{2 S_n \log\log S_n}} \leq 1.$$

Pf. Consider $Z_n := \dfrac{\exp X_n}{\prod_{j=1}^n \cosh(X_j - X_{j-1})}$ — exponential transform.

Assume $X_0 = 0$, take $Z_0 = 1$.

Note that $E(e^{(X_n - X_{n-1})} | \mathcal{F}_{n-1}) = \frac{1}{2}(e^{X_n - X_{n-1}} + e^{X_{n-1} - X_n}) = \cosh(X_n - X_{n-1})$.

Note also that $\cosh(X_n - X_{n-1})$ is $\mathcal{F}_{n-1}$-measurable (since $\cosh$ is an even function).

Therefore $E(Z_n | \mathcal{F}_{n-1}) = Z_{n-1} \dfrac{E(\exp(X_n - X_{n-1}))}{\cosh(X_n - X_{n-1})} = Z_{n-1}.$

So Z_n is a martingale.

In particular, $E(Z_n) = E(Z_0) = 1$.

Let $N_\alpha := \inf\{n : X_n \geq \alpha\}$ — stopping time.

Lemma. Let $E_{\alpha,\beta} = \{N_\alpha < \infty, S_{N_\alpha} \leq \beta\}$. Then

$P(E_{\alpha,\beta}) \leq e^{-\left(\frac{\alpha^2}{2\beta}\right)}$

Pf set $t := \alpha/\beta$, consider Z_n to be the exponential transform of tX_n. Then, since $\cosh \xi \leq e^{\xi^2/2}$, we have:

$Z_n \geq e^{tX_n} e^{-t^2 S_n/2}$. So

$$e^{\alpha^2/2\beta} P(E_{\alpha,\beta}) = E\left(\exp\left(t\alpha - \tfrac{t^2}{2}\beta\right) \chi_{E_{\alpha,\beta}}\right) \leq E(Z_{N_\alpha} \chi_{E_{\alpha,\beta}})$$

(since on $E_{\alpha,\beta}$, $Z_{N_\alpha} \geq \alpha$, $S_{N_\alpha} \leq \beta$). $\leq E(Z_{N_\alpha}) = E(Z_0) = 1$, optional stopping time. ∎

Now fix $\varepsilon > 0$.

Let $E := \{S = \infty, X_n > (1+\varepsilon)\sqrt{2 S_n \log\log S_n} \text{ i.o.}\}$.

We need: $|E| = 0$.

Define $T_k := \min\{n : S_n \geq (1+\varepsilon)^k\}$. Then

$E_k := \{S = \infty, \exists n \in [T_k, T_{k+1}) : X_n > (1+\varepsilon)\sqrt{2 S_n \log\log S_n}\}$

$E_k := \{ S = \infty,\ \exists\, n \in [T_k, T_{k+1}) : X_n > (1+\varepsilon)\sqrt{2 S_n \log\log S_n} \}$.

$E = \overline{\lim}\, E_k$, so need to show that $\sum P(E_k) < \infty$ and use Borel-Cantelli. But, by Lemma, since $S_n \le (1+\varepsilon)^{k+1}$, $X_n \ge (1+\varepsilon)\sqrt{2(1+\varepsilon)^k \log\log(1+\varepsilon)^k}$

$$P(E_k) \le \exp\left(-\frac{(1+\varepsilon)^2\, 2(1+\varepsilon)^k \log\log(1+\varepsilon)^k}{2(1+\varepsilon)^{k+1}}\right) = \left(\frac{1}{k\log(1+\varepsilon)}\right)^{1+\varepsilon}.$$

And the series $\sum \left(\frac{1}{k\log(1+\varepsilon)}\right)^{1+\varepsilon} < \infty$ — converges.