Prime ends

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Ω - bounded domain. (Remark: unbounded - the same in spherical metric).

Def Crosscut. $\gamma: [0,1] \to \overline{\Omega}$, $\gamma(0), \gamma(1) \in \partial\Omega$, $\gamma(0,1) \subset \Omega$.

Lemma (Koebe) $f: \mathbb{D} \to \Omega$ - conformal, γ - crosscut in $\Omega \Rightarrow f^{-1}(\gamma)$ is crosscut in $\mathbb{D}$.

Pf $f^{-1}(\gamma)$ - an arc in $\mathbb{D}$. Need $\lim_{t\to 0,1} f^{-1}(\gamma(t))$. Assume $\lim_{t\to 0} f^{-1}(\gamma(t))$ does not exist so $\exists\, t_n, t_n' \to 0$, $f^{-1}(\gamma(t_n)) \to z_1 \neq z_2 = f^{-1}(\gamma(t_n'))$.
By Lavrentiev (or extended version) $\lim_{n\to\infty} \operatorname{diam} \gamma(t_n, t_n') \geq c|z_1 - z_2|^p$. But of coarse $\operatorname{diam} \gamma(t_n, t_n') \to 0$ - contradiction.

Def. Chain $(\gamma_n, \mathcal{D}_n)$ in Ω: γ_n - crosscut in Ω, $\mathcal{D}_n$ - one of the components of $\Omega \setminus \gamma_n$, such that.

1) $\operatorname{diam} \gamma_n \to 0$ 2) $\mathcal{D}_n \subset \mathcal{D}_{n+1}$ 3) $\operatorname{dist}(\gamma_n, \gamma_{n+1}) > 0$.  Yes  No

Def. Two chains are equivalent, $(\gamma_n, \mathcal{D}_n) \sim (\gamma_n', \mathcal{D}_n')$ if $\forall n$ $\mathcal{D}_n$ contains all but finitely many $\mathcal{D}_m'$. Same for $\mathcal{D}_n'$.

Def. Prime end - equivalency class of crosscuts.

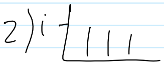
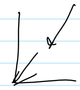
Lemma. Preimage of a chain under $f: \mathbb{D} \to \Omega$ is a chain.

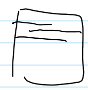
Pf. $f^{-1}(\gamma_n)$ - crosscut, $\operatorname{diam} f^{-1}(\gamma_n) \leq C\sqrt{\operatorname{diam} \gamma_n}$ - by our remark, $\operatorname{dist} > 0$ by Wolff,
$f^{-1}(\mathcal{D}_{n+1}) \subset f^{-1}(\mathcal{D}_n)$

Corollary. f^{-1}(Prime end) - prime end.

Def. P - prime end, support of prime end $I(P) := \bigcap \overline{\mathcal{D}_n}$ (does not depend on chain).

$I(P)$ - compact, connected

Examples: 0) $\mathbb{D}$, 1)  2)  $I(P) = [0, i]$. 3)  Uncountably many prime ends at 0.

4)  spiral $I(P) = \mathbb{T}$ 5)  $I(P) = [0,1]$ 6)  $I(P) = [0,1]$.

Thm (Caratheodory) $f: \mathbb{D} \to \Omega$. The correspondence $P \to \bigcap \overline{f^{-1}(\mathcal{D}_n)}$ is a bijection between prime ends and $\partial\mathbb{D}$.

Pf. 1) $f^{-1}(P)$ - prime end $\Rightarrow$ $\forall P\ \exists!$ point in $\mathbb{T}$, $\bigcap \overline{f^{-1}(\mathcal{D}_n)}$.

2) Injective: Let $P \to \zeta$, $P' \to \zeta \Rightarrow \left(f^{-1}(\gamma_n), f^{-1}(\mathcal{D}_n)\right) \sim \left(f^{-1}(\gamma_n'), f^{-1}(\mathcal{D}_n')\right) \overset{\text{take } f}{\Rightarrow}$ $(\gamma_n, \mathcal{D}_n) \sim (\gamma_n', \mathcal{D}_n')$.

3) Surjective. Take $\zeta \in \partial\mathbb{D}$. Take $z_1 \in \mathbb{D}$.
By Wolff, $\exists$ crosscut γ_1, such that $f^{-1}(\gamma)$ separates ζ and z_1 from 0, and $\operatorname{diam} \gamma_1 \leq \dfrac{C}{\sqrt{\log^+ \frac{1}{|z_1 - \zeta|}}}$. Let $d = \operatorname{dist}(\zeta, f^{-1}(\gamma_1))$. Take z_2 so that

 $\dfrac{C}{\sqrt{\log^+ \frac{1}{|z_2 - \zeta|}}} < \dfrac{d}{100}$ - Then γ_2 lies inside γ_1, $\operatorname{diam} f(\gamma_2) < \dfrac{C}{\sqrt{\log^+ \frac{1}{|z_2 - \zeta|}}}$.

Repeat to construct $\gamma_n \not\ni \zeta$, $\left(f(\gamma_n), f(\mathcal{D}_n)\right)$ - prime end. ∎

Corollary. $(\gamma_n, \mathcal{D}_n) \nsim (\gamma_n', \mathcal{D}_n') \iff \exists n: \mathcal{D}_n \cap \mathcal{D}_n' = \emptyset$.

Pf. Pull back to $\mathbb{D}$ ∎

Def. $\mathcal{P}(\Omega)$ – set of prime ends – Caratheodory boundary. $\hat{\Omega} := \Omega \cup \mathcal{P}(\Omega)$, $\hat{f}: \overline{\mathbb{D}} \to \hat{\Omega}$.
Mazurkiewicz metric extends to $\mathcal{P}(\Omega)$: shortest crosscut separating ...
As before, $c|z_1 - z_2|^2 \leq \rho(\hat{f}(z_1), \hat{f}(z_2)) \leq \dfrac{C}{\sqrt{\log \frac{1}{|z_1 - z_2|}}}$.

Thm (Caratheodory). Ω – Jordan domain, then conformal $f: \mathbb{D} \to \Omega$ can be extended to homeomorphism $\hat{f}: \overline{\mathbb{D}} \to \overline{\Omega}$.

Pf. Need to check:
1) Every prime end is a point: γ_n joins $\alpha(t_n)$ to $\alpha(t_n')$, α – homeo $\Rightarrow$ $|t_n - t_n'| \to 0$. $|\alpha(t_n) - \alpha(t_n')| \to 0$
So $\lim \alpha(t_n) = \lim \alpha(t_n') = I(P)$.
2) Every point is prime end: take $\zeta \in \partial\Omega$. D_n = component of $B(\zeta, \frac{1}{n}) \cap \Omega$ containing ζ at the boundary. $\bigcap D_n = \{\zeta\}$ ∎

Remark. Same way can prove: $f: \mathbb{D} \to \Omega$ extends to $f: \overline{\mathbb{D}} \to \overline{\Omega}$ iff $\partial\Omega$ is locally connected, i.e. $\forall \varepsilon > 0\ \exists \delta > 0$: $z_1, z_2 \in \partial\Omega$, $|z_1 - z_2| < \delta \Rightarrow \exists C \subset \partial\Omega$ $z_1, z_2 \in C$, C – connected, $\operatorname{diam} C < \varepsilon$.

Limit sets.

Def. $f: \mathbb{D} \to \mathbb{C}$ – a map (not necessarily analytic), $\zeta \in \partial\mathbb{D}$.
1) Full limit set at ζ: $C(f, \zeta) = \{a \in \hat{\mathbb{C}} : \exists z_n \to \zeta : f(z_n) \to a\} \Leftrightarrow$
$C(f, \zeta) = \bigcap_n \overline{f(B(\zeta, \frac{1}{n}) \cap \mathbb{D})}$.
2) γ – semicrosscut ending at ζ; $C_\gamma(f, \zeta)$ – limit set along γ $= \{a \in \hat{\mathbb{C}} : \exists z_n \to \zeta, z_n \in \gamma,$ $f(z_n) \to a\}$.
$C_\gamma(f, \zeta) = z \Rightarrow z$ is called asymptotic value of f at ζ.

Lemma. $\exists\ \gamma : C_\gamma(f, \zeta) = C(f, \zeta)$

Pf. Arrange $z_n \to \zeta$, $f(z_n) \to C(f, \zeta)$, join by curve ∎

3) $C_{rad}(f, \zeta) = \{a \in \hat{\mathbb{C}} : \exists r_n \to 1-, f(r_n \zeta) \to a\} = C_{[0, \zeta]}(f, \zeta)$.

Thm (Collingwood maximality thm). f – continuous on $\mathbb{D}$. Then $C_{rad}(f, \zeta) = C(f, \zeta)$ except on a set of first cathegory.

4) $C_\Delta(f, \zeta)$ – over angle, $C_{Stolz}(f, \zeta) = \bigcup_\Delta C_\Delta(f, \zeta)$.

Limit sets and prime ends.

Thm (Caratheodory) $f: \mathbb{D} \to \Omega$ – conformal, $\zeta \to \mathcal{P}(\Omega)$. Then $C(f, \zeta) = I(P)$ **Pf.** Def of prime end ∎

Def. Set of principal points of prime end P:
$\Pi(P) = \{w \in I(P) : \exists \underset{\text{chain}}{(\gamma_n)} \in P,\ \gamma_n \to w\}$. Examples.

Thm (Lindelöf). $\Pi(P) = C_{rad}(f, \zeta) = C_{Stolz}(f, \zeta) = \bigcap_{\gamma\text{ – semicrosscut to }\zeta} C_\gamma(f, \zeta)$

Pf. 1) $\Pi(P) = \bigcap_\gamma C_\gamma(f, \zeta)$
Let $w \in \Pi(P)$. Take any γ. $f(\gamma)$ intersect any crosscut ~~from~~ (γ_n). z_n: $f(z_n) \in \gamma \cap \gamma_n$, $z_n \to \zeta$, $f(z_n) \to w$.

Other direction: $w \in I(P) \setminus \Pi(P)$. Then $\exists r$: $B(w,r)$ contains no crosscuts from $P \Rightarrow$ $f^{-1}(\partial B(w,r))$ does not separate ζ from $0 \Rightarrow$ $\exists\ \gamma$ not intersecting $f^{-1}(B(w,r))$, γ - joins 0 to ζ. Then $w \notin C_\gamma(f,\zeta)$.

2) $C_{rad}(f,\zeta) = C_{Stolz}(f,\zeta)$. Let $z_n \underset{\Delta}{\to} \zeta$ $f(z_n) \to w$. Then $\rho(|z_n|\zeta, z_n) < C_\Delta$.

So $\rho(f(z_n), f(|z_n|\zeta)) < C_\Delta$. So $\text{dist}(f(z_n), f(|z_n|\zeta)) \to 0$, so $f(|z_n|\zeta) \to w$.

$w \in C_{rad}(f,\zeta)$. Moreover, for any non-tangential γ, $C_\gamma = C_{rad}$.

3) $\bigcap C_\gamma \subset C_{rad}$. Other direction: $w \in I(P) \setminus \Pi(P)$. As before, $\exists r$:

$f^{-1}(\partial B(w,r))$ does not separate 0 from ζ. So $\exists$ non-tangential γ joining 0 to ζ.

So $w \notin C_\gamma(f,\zeta) = C_{rad}(f,\zeta)$