Problem 1. Let m, n, k be natural numbers, $k \mid mn$, and $(m, n) = 1$. Show that there are natural numbers k_1, k_2, such that $k = k_1 k_2$, $k_1 \mid m$ and $k_2 \mid n$.

Solution. Let $k_1 = (m, k)$ and $k_2 = \frac{k}{k_1}$. Then $k_1 \mid m$, $k_2 \in \mathbb{N}$ and $k = k_1 k_2$. It remains to show $k_2 \mid n$. Since $(k_1 k_2) \mid (nm)$ and $k_1 \mid m$, we have $k_2 \mid (n \cdot \frac{m}{k_1})$. Thus in order to show $k_2 \mid n$ it suffices to show $(k_2, \frac{m}{k_1}) = 1$. Since $(k, m) = k_1$, by Bezout's lemma, there are integers a, b such that $ak + bm = k_1$. Dividing by k_1, we get $ak_2 + b\frac{m}{k_1} = 1$. It follows from Bezout's lemma that $(k_2, \frac{m}{k_1}) = 1$, as desired.

Problem 3. Given an integer n, find the largest integer e such that

$$2^e \mid n^2 + 1 \; . \tag{0.1}$$

Solution. First note that e may depend on n. In other words, we are **not** looking for the largest e such that *for all* n, (0.1) holds.
Recall that a square integer is either 0 or 1 mod 4. More specifically, for $n \in \mathbb{N}$, n^2 is 0 mod 4 (1 mod 4, resp.) if n is even (odd, resp.). Thus $n^2 + 1$ is congruent to 1 mod 4 when n is even and is congruent to 2 mod 4 when n is odd. In the first case $e = 0$ and in the latter $e = 1$.