Introduction to Real Analysis

Assignment 10, due November 29

Problem 1 of 5. Let $f(x)$ be an integrable function on $[a,\, b]$ and g is another function on $[a,\, b]$ such that the set

$$\{x \in [a,\, b] \,:\, f(x) \neq g(x)\}$$

is finite. Prove that g is also integrable on $[a,\, b]$ and

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Hint: Consider a partition P for which $U(f,P) - L(f,P) < \varepsilon/2$. Add to P a few small intervals surrounding the points where $f \neq g$.

Problem 2 of 5. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a function with the property that for any $k \in \mathbb{N}$ the set $\{x \in \mathbb{R} \,:\, |f(x)| \geq 1/k\}$ has no limit points. Prove that f is integrable on any interval $[a,\, b]$ and

$$\int_a^b f(x)\,dx = 0.$$

Hint: you have seen such functions in Problem 4, Assignment 5.

Problem 3 of 5. For a bounded function f on an interval $[a,\, b]$ and a subinterval $[c,\, d] \subseteq [a,\, b]$, define

$$\omega(f; c, d) := \sup \{|f(x) - f(y)| \,:\, x,\, y \in [c,\, d]\}\,.$$

Show that f is integrable on $[a, b]$ if and only if for any $\varepsilon > 0$ one can find a partition $P = \{x_0,\, x_1, \ldots x_n\}$ of $[a,\, b]$ such that

$$\sum_{k=1}^{n} \omega(f; x_{k-1}, x_k)(x_k - x_{k-1}) < \varepsilon$$

Problem 4 of 5. Let f be an increasing (not necessarily continuous) function on $[a,\, b]$. Show that f is integrable on $[a,\, b]$.
Hint: If f is not a constant, for a fixed $\varepsilon > 0$, take any partition P with $|P| < \frac{\varepsilon}{f(b) - f(a)}$.

Problem 5 of 5. Let us define a sequence of functions $(h_n(x))$ on $[0,\, 1]$ by

$$h_n(x) = \sum_{k=1}^{n} \chi_{[\frac{k-1}{n},\, \frac{k}{n}]},$$

where $\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ is a characteristic function of a set A.

(1) Show that the series

$$\sum_{n=1}^{\infty} 2^{-n} h_n(x)$$

converges uniformly on $[0,\, 1]$.

(2) Let

$$h(x) := \sum_{n=1}^{\infty} 2^{-n} h_n(x)$$

Show that h is discontinuous at every rational point of $[0, 1]$ and continuous at every irrational point of $[0, 1]$.
Hint: Where is the function $h_n(x)$ continuous? Can a sum of a function continuous at a point and a function discontinuous at the same point be continuous at that point?

(3) Show that h is integrable on $[0, 1]$ and compute its integral. Justify your calculation.