Introduction to Real Analysis

Assignment 2, due September 20

Problem 1 of 10. Let A be a nonempty bounded set. Let

$$B := \{x \in \mathbb{R} \,:\, x \text{ is a lower bound for } A\}\,; \quad C := \{x \in \mathbb{R} \,:\, x \text{ is an upper bound for } A\}\,.$$

Prove that B is bounded above, C is bounded bellow, and

$$\inf A = \sup B; \qquad \sup A = \inf C.$$

Can either B or C be bounded?

Problem 2 of 10. Let $A_1, A_2, A_3, \ldots$ be a collection of bounded nonempty sets.

(1) Show that $\bigcup_{k=1}^{n} A_k$ is also bounded. Compute its supremum and infimum in terms of suprema and infima of $\{A_k\}_{k=1}^{n}$.

(2) Give an example of an infinite collection of bounded nonempty sets $\{A_k\}_{k=1}^{\infty}$ such that $\bigcup_{k=1}^{\infty} A_k$ is nopt bounded above or bellow.

(3) Assuming now that $\bigcup_{k=1}^{\infty} A_k$ is also bounded, compute its supremum and infimum in terms of suprema and infima of $\{A_k\}_{k=1}^{\infty}$.

Problem 3 of 10. Let A be a nonempty bounded set, $c \in \mathbb{R}$. Let

$$cA := \{cx \,:\, x \in A\}\,.$$

Prove that cA is also bounded and compute its supremum and infimum.

Problem 4 of 10. Assume that $\inf A > \inf B$. Show that there is $\varepsilon > 0$ and $b \in B$, such that $b + \varepsilon$ is a lower bound for A.

Problem 5 of 10. Let $\sup A < \inf B$. Show that there is $\varepsilon > 0$ and $c \in \mathbb{R}$, such that $c + \varepsilon$ is a lower bound for B and $c - \varepsilon$ is an upper bound for A.

Problem 6 of 10. Give an example of a sequence of nested *open* intervals $\big((a_n,\, b_n)\big)_{n=1}^{\infty}$, $(a_{n+1},\, b_{n+1}) \subset (a_n,\, b_n)$, such that $\bigcap_{n=1}^{\infty}(a_n,\, b_n) = \emptyset$

Problem 7 of 10. Assume that the sequence a_n is *strictly increasing*, i.e. $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Assume also that the sequence b_n is *strictly decreasing*, i.e. $b_{n+1} < b_n$ and $a_n < b_n$ for all $n \in \mathbb{N}$. Prove that $\bigcap_{n=1}^{\infty}(a_n,\, b_n) \neq \emptyset$.
Hint: Let $I_n = [a_n,\, b_n]$ be a closed interval. Observe that $I_{n+1} \subset (a_n,\, b_n)$ and thus $\bigcap_{n=1}^{\infty} I_{n+1} \subset \bigcap_{n=1}^{\infty}(a_n,\, b_n)$.

Problem 8 of 10. A set $A \subset \mathbb{R}$ is called *dense* in $\mathbb{R}$ if for every real $x < y \in \mathbb{R}$ one can find $a \in A$ with $x < a < y$.

(1) Let B be an infinite subset of $\mathbb{N}$. Prove that the set of all rational numbers of the form $\dfrac{p}{q}$, where $p \in \mathbb{Z}$, $q \in B$, is dense in $\mathbb{R}$.

(2) Prove that the set of all rational numbers of the form $\dfrac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $239|p| > q$ is not dense in $\mathbb{R}$.

Problem 9 of 10. A sequence (a_n) is called *wrongverging* to a if

$$\forall \varepsilon \in \mathbb{R}\, \exists N \,:\, n > N \implies a - a_n < \varepsilon.$$

(1) Give an example of a wrongverging sequence.

(2) Prove that if a sequence wrongverges to some $a \in \mathbb{R}$ then it also wrongverges to any $x \in \mathbb{R}$.

Problem 10 of 10. Let (a_n) be a sequence of strictly positive numbers $a_n > 0$ converging to 0. Let (b_n) be a sequence of real numbers and $b \in \mathbb{R}$.

(1) Assume that $\lim_{n\to\infty} b_n = b$. Prove that

$$\forall k\, \exists N \,:\, n > N \implies |b - b_n| < a_k.$$

Hint: For a fixed k, a_k is just a positive number.

(2) Assume now that

$$\forall k\, \exists N \,:\, n > N \implies |b - b_n| < a_k.$$

Prove that $\lim_{n\to\infty} b_n = b$.
Hint: Fix $\varepsilon > 0$. Then you can always find $a_k < \varepsilon$. (Why?)

You just established that $\lim_{n\to\infty} b_n = b$ if and only if

$$\forall k\, \exists N \,:\, n > N \implies |b - b_n| < a_k.$$