Introduction to Real Analysis

Assignment 3, due October 4

Problem 1 of 5. Let (a_n) be a sequence of positive numbers. Let $b_n := \dfrac{a_n}{1+a_n}$

(1) Prove that the sequence b_n is bounded.

(2) Prove that if the sequence a_n is bounded then $\limsup b_n < 1$.

(3) Prove that if the sequence a_n is not bounded then $\limsup b_n = 1$.

(4) Prove that $\liminf b_n = 0$ if and only if $\liminf a_n = 0$.

Problem 2 of 5. Let (a_n) be a bounded sequence. Define

$$S := \{x \,:\, x < a_n \text{ for infinitely many } n\}.$$

Prove that S is bounded above and

$$\sup S = \limsup a_n.$$

Problem 3 of 5. Let (a_n) be a real sequence and $a \in \mathbb{R}$. Assume that every subsequence (a_{n_k}) of (a_n) contains a (sub-)subsequence $(a_{n_{k_l}})$ converging to a.

(1) Prove that the sequence (a_n) is bounded.

(2) Prove that (a_n) converges to a.

Problem 4 of 5.

(1) Let (a_n) be a sequence. Assume that the series $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ converges. Prove that the sequence (a_n) converges.

(2) The map $f : \mathbb{R} \mapsto \mathbb{R}$ is called a *contraction* if for some $q < 1$, and for all $x,\ y \in \mathbb{R}$,

$$|f(x) - f(y)| \leq q|x - y|.$$

Let the sequence (a_n) be defined recursively: $a_1 \in \mathbb{R}$, $a_{n+1} = f(a_n)$. Use the first part to show that the sequence (a_n) converges.

(3) Prove that $a := \lim_{n\to\infty} a_n$ is the unique *fixed point* of f, i. e. the unique $a \in \mathbb{R}$ with $f(a) = a$.

Problem 5 of 5 (*Raabe's Summability Test*). Assume that (a_n) is a positive sequence and

$$\liminf n\left(\frac{a_n}{a_{n+1}} - 1\right) > 1.$$

(1) Show that for some $\varepsilon > 0$ one can find N so that for $n \geq N$,

$$na_n - (n+1)a_{n+1} > \varepsilon a_{n+1}$$

(2) Let $s_n = \sum_{k=1}^{n} a_k$ be the sequence of partial sums. Sum up the inequalities in the previous part for $n = N+1, N+2, \ldots, N+p-1$ to obtain that for any $p \in \mathbb{N}$

$$\varepsilon(s_{N+p} - s_N) < Na_N - (N+p)a_{N+p} < Na_N.$$

(3) Prove that for any $p \in \mathbb{N}$:

$$s_{N+p} < s_N + \frac{N a_N}{\varepsilon}.$$

Conclude that the sequence (s_n) is bounded, and thus the series $\sum_{n=1}^{\infty} a_n$ converges.

(4) Let now

$$a_n := \left[\frac{1 \cdot 4 \cdot 7 \cdot \dots \cdot (3n-2)}{3 \cdot 6 \cdot 9 \cdot \dots \cdot 3n} \right]^2.$$

Use the previous part to show that the series $\sum_{n=1}^{\infty} a_n$ converges.