Introduction to Real Analysis

Assignment 4, due October 18

Problem 1 of 5. Let A be a set. Let

$$\operatorname{dist}(x, A) := \inf_{a \in A} |x - a|.$$

Let $U_n : \left\{x \,:\, \operatorname{dist}(x, A) < \frac{1}{n}\right\}$.

(1) Show that the set U_n is open. **Hint:** It can be represented as a union of open sets.

(2) Show that $\overline{A} = \cap_{n=1}^{\infty} U_n$.

Problem 2 of 5. Let A and B be subsets of $\mathbb{R}$.

(1) Prove that $\operatorname{Int}(A \bigcap B) = \operatorname{Int} A \bigcap \operatorname{Int} B$.

(2) Prove that $\operatorname{Int}(A \bigcup B) \subset \operatorname{Int} A \bigcup \operatorname{Int} B$.
Corrected on October 17: The inclusion should be $\operatorname{Int}(A \bigcup B) \supset \operatorname{Int} A \bigcup \operatorname{Int} B$. As explained in the announcement, this part of the problem is now optional.

(3) Give an example of two sets A and B with $\operatorname{Int}(A \bigcup B) \neq \operatorname{Int} A \bigcup \operatorname{Int} B$.

Problem 3 of 5.

(1) Prove that if A is a bounded above set then $\sup A \in \operatorname{Bd} A$.

(2) Prove that if A is a bounded bellow set then $\inf A \in \operatorname{Bd} A$.

(3) Prove that if $a < b < c$ and the two sets A and B has the property that $A \bigcap (a, c) = B \bigcap (a, c)$. Show that $b \in \operatorname{Bd} A$ if and only if $b \in \operatorname{Bd} B$.

(4) Prove that if $A \subset \mathbb{R}$ and $\operatorname{Bd} A = \emptyset$ then $A = \mathbb{R}$ or $A = \emptyset$. **Hint:** Consider $a \in A$, $c \in A^c$, and try to use the previous result to create a bounded set with no boundary points.

(5) Derive that the only subsets of $\mathbb{R}$ which are both open and closed are $\mathbb{R}$ itself and $\emptyset$.

Problem 4 of 5. Give counterexamples to the following false statements

(1) The isolated points of a set form a closed set.

(2) Every open set contains a t least two points.

(3) The supremum of a bounded above set is the greatest of its limit points.

(4) If A is any subset of $\mathbb{R}$, then $\operatorname{Bd} \overline{A} = \operatorname{Bd} A$.

(5) If A is any subset of $\mathbb{R}$, then $\operatorname{Bd} \operatorname{Bd} A = \operatorname{Bd} A$.

Problem 5 of 5. For $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, let

$$A + B := \{a + b \,:\, a \in A,\ b \in B\}.$$

Let A be a closed set, B be a compact set. Show that $A + B$ is closed.
Hint: Consider a limit point of c of $A + B$. Then $c = \lim_{n\to\infty}(a_n + b_n)$. Use a converging subsequence of (a_n).