Introduction to Real Analysis

Assignment 5, due October 25

Problem 1 of 5. Let $A \subset \mathbb{R}$ be a bounded set and $\varepsilon > 0$.

(1) Consider the collection $(V_\varepsilon(a))_{a\in A}$. Prove that it is an open cover of $\overline{A}$.

(2) Conclude that there is a finite subcover $(V_\varepsilon(a_j))_{j=1}^n$: $A \subset \bigcup_{j=1}^n V_\varepsilon(a_j)$.

(3) A subset $B \subset A$ is called ε-net if for any $a \in A$ one can find $b \in B$ such that $|a - b| < \varepsilon$. Use the previous part to show that there exists a finite ε-net.

Problem 2 of 5. Let $A = \{a_1,\ a_2, \dots, a_n, \dots\} \subset \mathbb{R}$ be a countable set dense in $\mathbb{R}$.

(1) Let U be an open set, $A \subset U$. Show that U^c is nowhere dense.

(2) Assume that A can be represented as $A = \bigcap_{n=1}^\infty U_n$, where all U_n are open. Let $F_n := \{a_n\}$ be closed sets containing one point each. Show that

$$\mathbb{R} = (\cup_{n=1}^\infty U_n^c) \bigcup (\cup_{n=1}^\infty F_n)\,.$$

(3) Arrive to a contradiction with Baire's Theorem, thus proving that A cannot be represented as $A = \bigcap_{n=1}^\infty U_n$, where all U_n are open.

Problem 3 of 5. A function $f : A \mapsto \mathbb{R}$ is called *bounded* near a, a limit point of A, if for some M and $\delta > 0$, if $0 < |x - a| < \delta$ and $x \in A$ then $|f(x)| \le M$.

(1) Show that if $\lim_{x\to a} f(x)$ exists then f is bounded near a.

(2) Give an example of a function f bounded near a point a such that $\lim_{x\to a} f(x)$ does not exist.

(3) Let $f, g : A \mapsto \mathbb{R}$ be two functions on a set A with a limit point a. Let f be bounded near a and $\lim_{x\to a} g(x) = 0$. Prove that $\lim_{x\to a} f(x)g(x) = 0$.

Problem 4 of 5.

(1) Let $g : \mathbb{R} \mapsto \mathbb{R}$ be a function with the property that for any $k \in \mathbb{N}$ the set $\{x \in \mathbb{R} : |g(x)| \ge 1/k\}$ has no limit points. Show that for any $k \in \mathbb{N}$ and for any $a \in \mathbb{R}$ there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|g(x)| < \frac{1}{k}$.

(2) Prove that for any $a \in \mathbb{R}$, $\lim_{x\to a} g(x) = 0$.

(3) Let now

$$f(x) = \begin{cases} \frac{1}{\min\{n\in\mathbb{N} :\, nx\in\mathbb{Z}\}}, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Using the previous part, prove that for any $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = 0$.

(4) Describe the points of continuity of f.

Problem 5 of 5. Let f be a continuous function on a closed set A and K be a closed set.

(1) Let $(a_n) \subset A$ be a sequence of points of A converging to a point a. Assume that for all $n \in \mathbb{N}$, $f(a_n) \in K$. Prove that $f(a) \in K$.

(2) Prove that the set $f^{-1}(K) := \{a \in A : f(a) \in K\}$ is a closed subset of A.
Hint: Let a be a limit point of $f^{-1}(K)$. Use the previous part to prove that $a \in f^{-1}(K)$.