Introduction to Real Analysis

Assignment 6, due November 1

Problem 1 of 5. Let $A \subset \mathbb{R}$. Define the *characteristic function* of A as

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

(1) Show that χ_A is discontinuous for $x \in \operatorname{Bd} A$ and continuous at all other points.

(2) Show that the function $f(x) := \operatorname{dist}(x, A)\chi_{\mathbb{Q}}(x)$ is continuous on $\overline{A}$ and discontinuous at all other points.

(3) Show that the function $g(x) := \operatorname{dist}(x, A)\chi_{\mathbb{Q}}(x) + \chi_A$ is continuous on $\operatorname{Int} A$ and discontinuous at all other points.

Problem 2 of 5.

(1) Let $A \subset \mathbb{R}$ be a bounded set. Show that every function uniformly continuous on A is bounded.
Hint: You can use Problem 1, part (3) of Assignment 5.

(2) Let f be a function uniformly continuous on two sets A and B. Show that f is uniformly continuous on $A \cap B$.

(3) Let $B \subset A$ and f be uniformly continuous on A. Show that f is also uniformly continuous on B.

Problem 3 of 5. For this problem, you can use the fact that if $0 < x < y$ and $\alpha > 0$, then for some c with $x < c < y$ we have

$$x^\alpha - y^\alpha = \alpha(x - y)c^{\alpha - 1}$$

(1) Show that $f(x) = x^\alpha$ is uniformly continuous on $[1,\ \infty)$ if and only if $\alpha \leq 1$.

(2) Show that $f(x) = x^\alpha$ is uniformly continuous on $(0,\ 1]$ if and only if $\alpha \geq 0$.

(3) Show that $f(x) = x^\alpha$ is uniformly continuous on $(0,\ \infty)$ if and only if $0 \leq \alpha \leq 1$.

Problem 4 of 5.

(1) Let f be a function uniformly continuous on a set A, and g be a function uniformly continuous on a set B such that $f(A) \subset B$. Show that the function $g(f(x))$ is also uniformly continuous on A.

(2) Construct a set $A \subset \mathbb{R}$, a function f uniformly continuous on A and a function g *not* uniformly continuous on $B = f(A)$ such that the function $g(f(x))$ is still uniformly continuous on A.
Hint: You can use the result of Problem 3.

Problem 5 of 5. Assume that f is a function on the interval $[1,\ 2]$ which takes only rational values on the interval $[0,\ 1)$ and only irrational values on the interval $(1,\ 2]$. Assume also that f has finitely many points of discontinuity. Show that such an f is always discontinuous at exactly one point. Find this point.
Hint: Use Intermediate Value Theorem.