Introduction to Real Analysis

Assignment 7 (Midterm review), due November 8

Problem 1 of 5. Let $(I_n)_{n\in\mathbb{N}}$, $I_n = [a_n, b_n]$ be a sequence of nested closed intervals $I_{n+1} \subset I_n$, $n \in \mathbb{N}$.

(1) Prove that $\bigcap_{n\in\mathbb{N}} I_n$ is either a single point or a closed interval.

(2) Prove that $\bigcap_{n\in\mathbb{N}} I_n$ is a single point if and only if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(3) Let $\bigcap_{n\in\mathbb{N}} I_n$ be a single point, and let $x_n \in I_n$ for every $n \in \mathbb{N}$. Prove that the sequence (x_n) converges.

Problem 2 of 5. A sequence (a_n) is called *quasi-decreasing* if $\forall \varepsilon > 0\ \exists N$: if $n > m \geq N$ then $a_n < a_m + \varepsilon$.

(1) Show that a quasi-decreasing sequence is always bounded above.

(2) Is every bounded quasi-decreasing sequence converges? Prove or give a counter-example.

Problem 3 of 5.

(1) Show that the Cantor set C, defined on pages 85-86 of the textbook, is compact.

(2) Show that this set is nowhere dense.

(3) Let (a_n) be a fixed sequence of real numbers. Show that there exists $x \in \mathbb{R}$ which cannot be represented as a sum

$$x = a_n + c; \quad c \in C.$$

Hint: For each n, consider the set $a_n + C := \{x : x = a_n + c,\ c \in C\}$. Show that each of them is nowhere dense and use Baire's Theorem.

Problem 4 of 5. Let f be a continuous function on closed interval $[a, b]$.

(1) Show that $f([a, b])$ is a closed interval.

(2) Show that if f is injective then the inverse function f^{-1} is continuous on $f([a, b])$.
Hint: Let a sequence $(y_n) \subset f([a, b])$ converges to $y \in f([a, b])$. Let $y_n = f(x_n)$, $y = f(x)$. Show that any converging subsequence of (x_n) converges to x and conclude that (x_n) itself converges to x. Now remember that $x_n = f^{-1}(y_n)$, $x = f^{-1}(y)$.

Problem 5 of 5. Let f be an increasing continuous function on a closed interval $[a, b]$, differentiable on (a, b), $f(a) > a$, $f(b) < b$, and such that for all $x \in (a, b)$, $f'(x) \neq 1$.

(1) Show that for all $x \in (a, b)$, $0 \leq f'(x) < 1$.
Hint: First, use the Mean Value Theorem (for the points a and b) to show that for *some* $x \in (a, b)$, $f'(x) < 1$. Then use Darboux's theorem to show that it is true for *all* $x \in (a, b)$.

(2) Prove that f has exactly one *fixed point* $c \in (a, b)$, i.e. a point where $f(c) = c$.
Hint: For the existence, apply Intermediate Value Theorem to the function $f(x)-x$. For the uniqueness, use Mean Value Theorem and the fact that $f'(x) \neq 1$.

(3) Let $x_1 \in [a, b]$ and let the sequence x_n be defined recursively by $x_{n+1} = f(x_n)$. Show that $\lim_{n\to\infty} x_n = c$.
Hint: Show that the sequence (x_n) is increasing if $x_1 \leq c$ and decreasing if if $x_1 \geq c$. Use the continuity of f to show that $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} x_n$.